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# 1 INTRODUCTION

In the attempt to unify all the forces present in Nature, which entails having a consistent quantum theory of gravity, superstring theories seem promising candidates. The evidence that string theories could be unified theories is provided by the presence in their massless spectrum of enough particles to account for those present at low energies, including the graviton [1].

String theory is by now a vast subject with almost three decades of active research contributing to its development. During the last few years, our understanding of string theory has undergone a dramatic change. The key to this development is the discovery of duality symmetries, which relate the strong and weak coupling limits of different string theories ( $S$ -duality). These symmetries not only relate apparently different string theories, but give us a way to compute certain strong coupling results in one string theory by mapping it to a weak coupling result in a dual string theory (for recent review on non-perturbative string theory see for instance [2]).

Our main aim here is the description and further investigation of the properties of the so called null or tensionless branes. To explain how these extended objects appear and why it is worthwhile to learn more about them, we first will give a brief introduction to the related notions in string theory following mainly [3].

## 1.1 Brief Overview of String Theory

String theory is a description of dynamics of objects with one spatial direction, which we parameterize by  $\sigma$ , propagating in a space parameterized by  $x^\mu$ . The world-sheet of the string is parameterized by coordinates  $(\tau, \sigma)$  where each  $\tau = \text{constant}$  denotes the string at a given time. The amplitude for propagation of a string from an initial configuration to a final one is given by sum over world-sheets which interpolate

between the two string configurations weighed by  $\exp(iS)$ , where

$$S \propto \int d\tau d\sigma \partial_J x^\mu \partial^J x^\nu g_{\mu\nu}(x) \quad (1.1)$$

where  $g_{\mu\nu}$  is the metric on space-time and  $J$  runs over the  $\tau$  and  $\sigma$  directions. Note that by slicing the world-sheet we will get configurations where a single string splits to a pair or vice versa, and combinations thereof.

If we consider propagation in flat space-time where  $g_{\mu\nu} = \eta_{\mu\nu}$  the fields  $x^\mu$  on the world-sheet, which describe the position in space-time of each bit of string, are free fields and satisfy the 2 dimensional equation

$$\partial_J \partial^J x^\mu = (\partial_\tau^2 - \partial_\sigma^2) x^\mu = 0.$$

The solution of which is given by

$$x^\mu(\tau, \sigma) = x_L^\mu(\tau + \sigma) + x_R^\mu(\tau - \sigma).$$

In particular notice that the left- and right-moving degrees of freedom are essentially independent. There are two basic types of strings: *Closed* strings and *Open* strings depending on whether the string is a closed circle or an open interval respectively. If we are dealing with closed strings the left- and right-moving degrees of freedom remain essentially independent but if are dealing with open strings the left-moving modes reflecting off the left boundary become the right-moving modes—thus the left- and right-moving modes are essentially identical in this case. In this sense an open string has ‘half’ the degrees of freedom of a closed string and can be viewed as a ‘folding’ of a closed string so that it looks like an interval.

There are two basic types of string theories, bosonic and fermionic. What distinguishes bosonic and fermionic strings is the existence of supersymmetry on the world-sheet. This means that in addition to the coordinates  $x^\mu$  we also have anti-commuting fermionic coordinates  $\psi_{L,R}^\mu$  which are space-time vectors but fermionic spinors on the worldsheet whose chirality is denoted by subscript  $L, R$ . The action for superstrings takes the form

$$S = \int \partial_L x^\mu \partial_R x^\mu + \psi_R^\mu \partial_L \psi_R^\mu + \psi_L^\mu \partial_R \psi_R^\mu.$$

There are two consistent boundary conditions on each of the fermions, periodic (**R**amond sector) or anti-periodic (**N**eveu-**S**chwarz sector) (note that the coordinate  $\sigma$  is periodic).

A natural question arises as to what metric we should put on the world-sheet. In the above we have taken it to be flat. However in principle there is one degree of freedom that a metric can have in two dimensions. This is because it is a  $2 \times 2$  symmetric matrix (3 degrees of freedom) which is defined up to arbitrary reparametrization of 2 dimensional space-time (2 degrees of freedom) leaving us with one function. Locally we can take the 2 dimensional metric  $g_{JK}$  to be conformally flat

$$g_{JK} = \exp(\phi)\eta_{JK}.$$

Classically the action  $S$  does not depend on  $\phi$ . This is easily seen by noting that the properly coordinate invariant action density goes as  $\sqrt{|g|}g^{JK}\partial_J x^\mu\partial_K x^\nu\eta_{\mu\nu}$  and is independent of  $\phi$  only in  $D = 2$ . This is rather nice and means that we can ignore all the local dynamics associated with gravity on the world-sheet. This case is what is known as the critical string case which is the case of most interest. It turns out that this independence from the local dynamics of the world-sheet metric survives quantum corrections only when the dimension of space is 26 in the case of *bosonic strings* and 10 for *fermionic* or *superstrings*. Each string can be in a specific vibrational mode which gives rise to a particle. To describe the totality of such particles it is convenient to go to ‘light-cone’ gauge. Roughly speaking this means that we take into account that string vibration along their world-sheet is not physical. In particular for bosonic string the vibrational modes exist only in 24 transverse directions and for superstrings they exist in 8 transverse directions.

Solving the free field equations for  $x, \psi$  we have

$$\begin{aligned}\partial_L x^\mu &= \sum_n \alpha_{-n}^\mu e^{-in(\tau+\sigma)} \\ \psi_L^\mu &= \sum_n \psi_{-n}^\mu e^{-in(\tau+\sigma)}\end{aligned}\tag{1.2}$$

and similarly for right-moving oscillator modes  $\tilde{\alpha}_{-n}^\mu$  and  $\tilde{\psi}_{-n}^\mu$ . The sum over  $n$  in the above runs over integers for the  $\alpha_{-n}$ . For fermions depending on whether we are in

the **R** sector or **NS** sector it runs over integers or integers shifted by 1/2 respectively. Many things decouple between the left- and right-movers in the construction of a single string Hilbert space and we sometimes talk only about one of them. For the open string Fock space the left- and right-movers mix as mentioned before, and we simply get one copy of the above oscillators.

A special role is played by the *zero modes* of the oscillators. For the  $x$ -fields they correspond to the center of mass motion and thus  $\alpha_0$  gets identified with the left-moving momentum of the center of mass. In particular we have for the center of mass

$$x = \alpha_0(\tau + \sigma) + \tilde{\alpha}_0(\tau - \sigma),$$

where we identify

$$(\alpha_0, \tilde{\alpha}_0) = (P_L, P_R).$$

Note that for closed string, periodicity of  $x$  in  $\sigma$  requires that  $P_L = P_R = P$  which we identify with the center of mass momentum of the string.

In quantizing the fields on the strings we use the usual (anti)commutation relations

$$\begin{aligned} [\alpha_n^\mu, \alpha_m^\nu] &= n\delta_{m+n,0}\eta^{\mu\nu} \\ \{\psi_n^\mu, \psi_m^\nu\} &= \eta^{\mu\nu}\delta_{m+n,0}. \end{aligned}$$

We choose the negative moded oscillators as creation operators. In constructing the Fock space we have to pay special attention to the zero modes. The zero modes of  $\alpha$  should be diagonal in the Fock space and we identify their eigenvalue with momentum. For  $\psi$  in the *NS* sector there is no zero mode so there is no subtlety in construction of the Hilbert space. For the *R* sector, we have zero modes. In this case the zero modes form a Clifford algebra

$$\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu}.$$

This implies that in these cases the ground state is a spinor representation of the Lorentz group. Thus a typical element in the Fock space looks like

$$\alpha_{-n_1}^{L\mu_1} \dots \psi_{-n_k}^{L\mu_k} \dots |P_L, a\rangle \otimes \alpha_{-m_1}^{R\mu_1} \dots \psi_{-m_r}^{R\mu_r} \dots |P_R, b\rangle,$$

where  $a, b$  label spinor states for R sectors and are absent in the NS case; moreover for the bosonic string we only have the left and right bosonic oscillators.

It is convenient to define the total oscillator number as sum of the negative oscillator numbers, for left- and right- movers separately.  $N_L = n_1 + \dots + n_k + \dots$ ,  $N_R = m_1 + \dots + m_r + \dots$ . The condition that the two dimensional gravity decouple implies that the energy momentum tensor annihilate the physical states. The trace of the energy momentum tensor is zero here (and in all compactifications of string theory) and so we have two independent components which can be identified with the left- and right- moving hamiltonians  $H_{L,R}$  and the physical states condition requires that

$$H_L = N_L + (1/2)P_L^2 - \delta_L = 0 = H_R = N_R + (1/2)P_R^2 - \delta_R, \quad (1.3)$$

where  $\delta_{L,R}$  are normal ordering constants which depend on which string theory and which sector we are dealing with. For bosonic string  $\delta = 1$ , for superstrings we have two cases: For *NS* sector  $\delta = 1/2$  and for the *R* sector  $\delta = 0$ . The equations (1.3) give the spectrum of particles in the string perturbation theory. Note that  $P_L^2 = P_R^2 = -m^2$  and so we see that  $m^2$  grows linearly with the oscillator number  $N$ , up to a shift:

$$(1/2)m^2 = N_L - \delta_L = N_R - \delta_R. \quad (1.4)$$

### 1.1.1 Massless States of Bosonic Strings

Let us consider the left-mover excitations. Since  $\delta = 1$  for bosonic string, (1.4) implies that if we do not use any string oscillations, the ground state is tachyonic  $1/2m^2 = -1$ . This clearly implies that bosonic string by itself is not a good starting point for perturbation theory. Nevertheless in anticipation of a modified appearance of bosonic strings in the context of heterotic strings, let us continue to the next state.

If we consider oscillator number  $N_L = 1$ , from (1.4) we learn that excitation is massless. Putting the right-movers together with it, we find that it is given by

$$\alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu |P\rangle.$$



What is the physical interpretation of these massless states? The most reliable method is to find how they transform under the little group for massless states which in this case is  $SO(24)$ . If we go to the light cone gauge, and count the physical states, which roughly speaking means taking the indices  $\mu$  to go over spatial directions transverse to a null vector, we can easily deduce the content of states. By decomposing the above massless state under the little group of  $SO(24)$ , we find that we have symmetric traceless tensor, anti-symmetric 2-tensor, and the trace, which we identify as arising from 26 dimensional fields

$$g_{\mu\nu}, B_{\mu\nu}, \phi \tag{1.5}$$

the metric, the anti-symmetric field  $B$  and the *dilaton*. This triple of fields should be viewed as the stringy multiplet for gravity. The quantity  $\exp[-\phi]$  is identified with the string coupling constant. What this means is that a world-sheet configuration of a string which sweeps a genus  $g$  curve, which should be viewed as  $g$ -th loop correction for string theory, will be weighed by  $\exp(-2(g-1)\phi)$ . The existence of the field  $B$  can also be understood (and in some sense predicted) rather easily. If we have a point particle it is natural to have it charged under a gauge field, which introduces a term  $\exp(i \int A)$  along the world-line. For strings the natural generalization of this requires an anti-symmetric 2-form to integrate over the world-sheet, and so we say that the strings are *charged* under  $B_{\mu\nu}$  and that the amplitude for a world-sheet configuration will have an extra factor of  $\exp(i \int B)$ .

Since bosonic string has tachyons we do not know how to make sense of that theory by itself.

### 1.1.2 Massless States of Type II Superstrings

Let us now consider the light particle states for superstrings. We recall from the above discussion that there are two sectors to consider, NS and R, separately for the left- and the right-movers. As usual we will first treat the left- and right-moving sectors separately and then combine them at the end. Let us consider the NS sector for left-movers. Then the formula for masses (1.3) implies that the ground state

is tachyonic with  $1/2m^2 = -1/2$ . The first excited states from the left-movers are massless and corresponds to  $\psi_{-1/2}^\mu|0\rangle$ , and so is a vector in space-time. How do we deal with the tachyons? It turns out that summing over the boundary conditions of fermions on the world-sheet amounts to keeping the states with a fixed fermion number  $(-1)^F$  on the world-sheet. Since in the NS sector the number of fermionic oscillator correlates with the integrality/half-integrality of  $N$ , it turns out that the consistent choice involves keeping only the  $N = \text{half-integral}$  states. This is known as the GSO projection. Thus the tachyon is projected out and the lightest left-moving state is a massless vector.

For the R-sector using (1.3) we see that the ground states are massless. As discussed above, quantizing the zero modes of fermions implies that they are spinors. Moreover GSO projection, which is projection on a definite  $(-1)^F$  state, amounts to projecting to spinors of a given chirality. So after GSO projection we get a massless spinor of a definite chirality. Let us denote the spinor of one chirality by  $s$  and the other one by  $s'$ .

Now let us combine the left- and right-moving sectors together. Here we run into two distinct possibilities: A) The GSO projections on the left- and right-movers are different and lead in the R sector to ground states with different chirality. B) The GSO projections on the left- and right-movers are the same and lead in the R sector to ground states with the same chirality. The first case is known as type IIA superstring and the second one as type IIB. Let us see what kind of massless modes we get for either of them. From  $NS \otimes NS$  we find for both type IIA,B

$$NS \otimes NS \rightarrow v \otimes v \rightarrow (g_{\mu\nu}, B_{\mu\nu}, \phi)$$

From the  $NS \otimes R$  and  $R \otimes NS$  we get the fermions of the theory (including the gravitinos). However the IIA and IIB differ in that the gravitinos of IIB are of the same chirality, whereas for IIA they are of the opposite chirality. This implies that IIB is a chiral theory whereas IIA is non-chiral. Let us move to the  $R \otimes R$  sector. We find

$$IIA : R \otimes R = s \otimes s' \rightarrow (A_\mu, C_{\mu\nu\rho})$$

$$IIB : R \otimes R = s \otimes s \rightarrow (\chi, B'_{\mu\nu}, D_{\mu\nu\rho\lambda}), \quad (1.6)$$

where all the tensors appearing above are fully antisymmetric. Moreover  $D_{\mu\nu\rho\lambda}$  has a self-dual field strength  $F = dD = *F$ . It turns out that to write the equations of motion in a unified way it is convenient to consider a generalized gauge fields  $\mathcal{A}$  and  $\mathcal{B}$  in the IIA and IIB case respectively by adding all the fields in the RR sector together with the following properties: i)  $\mathcal{A}(\mathcal{B})$  involve all the odd (even) dimensional antisymmetric fields. ii) the equation of motion is  $d\mathcal{A} = *\mathcal{A}$ . In the case of all fields (except  $D_{\mu\nu\lambda\rho}$ ) this equation allows us to solve for the forms with degrees bigger than 4 in terms of the lower ones and moreover it implies the field equation  $d * d\mathcal{A} = 0$  which is the familiar field equation for the gauge fields. In the case of the  $D$ -field it simply gives that its field strength is self-dual.

### 1.1.3 Open Superstring: Type I String

In the case of type IIB theory in 10 dimensions, we note that the left- and right-moving degrees of freedom on the worldsheet are the same. In this case we can ‘mod out’ by a reflection symmetry on the string; this means keeping only the states in the full Hilbert space which are invariant under the left-/right-moving exchange of quantum numbers. This is simply projecting the Hilbert space onto the invariant subspace of the projection operator  $P = \frac{1}{2}(1 + \Omega)$  where  $\Omega$  exchanges left- and right-movers.  $\Omega$  is known as the orientifold operation as it reverses the orientation on the world-sheet. Note that this symmetry only exists for IIB and not for IIA theory (unless we accompany it with a parity reflection in spacetime). Let us see which bosonic states we will be left with after this projection. From the NS-NS sector  $B_{\mu\nu}$  is odd and projected out and thus we are left with the symmetric parts of the tensor product

$$NS - NS \rightarrow (v \otimes v)_{symm.} = (g_{\mu\nu}, \phi).$$

From the R-R sector since the degrees of freedom are fermionic from each sector we get, when exchanging left- and right-movers an extra minus sign which thus means

we have to keep anti-symmetric parts of the tensor product

$$R - R \rightarrow (s \otimes s)_{anti-symm.} = \tilde{B}_{\mu\nu}.$$

This is not the end of the story, however. In order to make the theory consistent we need to introduce a new sector in this theory involving open strings. This comes about from the fact that in the R-R sector there actually is a 10 form gauge potential which has no propagating degree of freedom, but acquires a tadpole. Introduction of a suitable open string sector cancels this tadpole.

As noted before the construction of open string sector Hilbert space proceeds as in the closed string case, but now, the left-moving and right-moving modes become indistinguishable due to reflection off the boundaries of open string. We thus get only one copy of the oscillators. Moreover we can associate ‘Chan-Paton’ factors to the boundaries of open string. To cancel the tadpole it turns out that we need 32 Chan-Paton labels on each end. We still have two sectors corresponding to the NS and R sectors. The NS sector gives a vector field  $A_\mu$  and the R sector gives the gaugino. The gauge field  $A_\mu$  has two additional labels coming from the end points of the open string and it turns out that the left-right exchange projection of the type IIB theory translates to keeping the antisymmetric component of  $A_\mu = -A_\mu^T$ , which means we have an adjoint of  $SO(32)$ . Thus all put together, the bosonic degrees of freedom are

$$(g_{\mu\nu}, \tilde{B}_{\mu\nu}, \phi) + (A_\mu)_{SO(32)}.$$

We should keep in mind here that  $\tilde{B}$  came not from the NS-NS sector, but from the R-R sector.

#### 1.1.4 Heterotic Strings

Heterotic string is a combination of bosonic string and superstring, where roughly speaking the left-moving degrees of freedom are as in the bosonic string and the right-moving degrees of freedom are as in the superstring. It is clear that this makes sense for the construction of the states because the left- and right-moving

sectors hardly talk with each other. This is almost true, however they are linked together by the zero modes of the bosonic oscillators which give rise to momenta  $(P_L, P_R)$ . Previously we had  $P_L = P_R$  but now this cannot be the case because  $P_L$  is 26 dimensional but  $P_R$  is 10 dimensional. It is natural to decompose  $P_L$  to a 10+16 dimensional vectors, where we identify the 10 dimensional part of it with  $P_R$ . It turns out that for the consistency of the theory the extra 16 dimensional component should belong to the root lattice of  $E_8 \times E_8$  or a  $Z_2$  sublattice of  $SO(32)$  weight lattice. In either of these two cases the vectors in the lattice with  $(length)^2 = 2$  are in one to one correspondence with non-zero weights in the adjoint of  $E_8 \times E_8$  and  $SO(32)$  respectively. These can also be conveniently represented (through bosonization) by 32 fermions: In the case of  $E_8 \times E_8$  we group them to two groups of 16 and consider independent NS, R sectors for each group. In the case of  $SO(32)$  we only have one group of 32 fermions with either NS or R boundary conditions.

Let us tabulate the massless modes using (1.4). The right-movers can be either NS or R. The left-moving degrees of freedom start out with a tachyonic mode. But (1.4) implies that this is not satisfying the level-matching condition because the right-moving ground state is at zero energy. Thus we should search on the left-moving side for states with  $L_0 = 0$  which means from (1.4) that we have either  $N_L = 1$  or  $(1/2)P_L^2 = 1$ , where  $P_L$  is an internal 16 dimensional vector in one of the two lattices noted above. The states with  $N_L = 1$  are

$$16 \oplus v,$$

where 16 corresponds to the oscillation direction in the extra 16 dimensions and  $v$  corresponds to vector in 10 dimensional spacetime. States with  $(1/2)P_L^2 = 1$  correspond to the non-zero weights of the adjoint of  $E_8 \times E_8$  or  $SO(32)$  which altogether correspond to 480 states in both cases. The extra 16  $N_L = 1$  modes combine with these 480 states to form the adjoints of  $E_8 \times E_8$  or  $SO(32)$  respectively. The right-movers give, as before, a  $v \oplus s$  from the NS and R sectors respectively. So putting the left- and right-movers together we finally get for the massless modes

$$(v \oplus Adj) \otimes (v \oplus s).$$

Thus the bosonic states are  $(v \oplus Adj) \otimes v$  which gives

$$(g_{\mu\nu}, B_{\mu\nu}, \phi; A_\mu),$$

where the  $A_\mu$  is in the adjoint of  $E_8 \times E_8$  or  $SO(32)$ . Note that in the  $SO(32)$  case this is an *identical* spectrum to that of type I strings.

### 1.1.5 Summary

To summarize, we have found 5 consistent strings in 10 dimensions: Type IIA with  $N = 2$  non-chiral supersymmetry, type IIB with  $N = 2$  chiral supersymmetry, type I with  $N=1$  supersymmetry and gauge symmetry  $SO(32)$  and heterotic strings with  $N=1$  supersymmetry with  $SO(32)$  or  $E_8 \times E_8$  gauge symmetry. Note that as far as the massless modes are concerned we only have four inequivalent theories, because heterotic  $SO(32)$  theory and Type I theory have the same light degrees of freedom. In discussing compactifications it is sometimes natural to divide the discussion between two cases depending on how many supersymmetries we start with. In this context we will refer to the type IIA and B as  $N = 2$  *theories* and Type I and heterotic strings as  $N = 1$  *theories*.

## 1.2 String Compactifications

So far we have only talked about superstrings propagating in 10 dimensional Minkowski spacetime. If we wish to connect string theory to the observed four dimensional spacetime, somehow we have to get rid of the extra 6 directions. One way to do this is by assuming that the extra 6 dimensions are tiny and thus unobservable in the present day experiments. In such scenarios we have to understand strings propagating not on ten dimensional Minkowski spacetime but on four dimensional Minkowski spacetime times a compact 6 dimensional manifold  $K$ . In order to gain more insight it is convenient to consider compactifications not just to 4 dimensions but to arbitrary dimensional spacetimes, in which case the dimension of  $K$  is variable.

The choice of  $K$  and the string theory we choose to start in 10 dimensions will lead to a large number of theories in diverse dimensions, which have different number of supersymmetries and different low energy effective degrees of freedom. In order to get a handle on such compactifications it is useful to first classify them according to how much supersymmetry they preserve. This is useful because the higher the number of supersymmetry the less the quantum corrections there are.

If we consider a general manifold  $K$  we find that the supersymmetry is completely broken. This is the case we would really like to understand, but it turns out that string perturbation theory always breaks down in such a situation; this is intimately connected with the fact that typically cosmological constant is generated by perturbation theory and this destabilizes the Minkowski solution. For this reason we do not even have a single example of such a class whose dynamics we understand. Instead if we choose  $K$  to be of a special type we can preserve a number of supersymmetries.

For this to be the case, we need  $K$  to admit some number of covariantly constant spinors. This is the case because the number of supercharges which are ‘unbroken’ by compactification is related to how many covariantly constant spinors we have. To see this note that if we wish to define a *constant* supersymmetry transformation, since a space-time spinor, is also a spinor of internal space, we need in addition a constant spinor in the internal compact directions. The basic choices are manifolds with trivial holonomy (flat tori are the only example),  $SU(n)$  holonomy (Calabi-Yau  $n$ -folds),  $Sp(n)$  holonomy ( $4n$  dimensional manifolds), 7-manifolds of  $G_2$  holonomy and 8-manifolds of  $Spin(7)$  holonomy. The case mostly studied in physics involves toroidal compactification,  $SU(2) = Sp(1)$  holonomy manifold (the 4-dimensional  $K3$ ),  $SU(3)$  holonomy (Calabi-Yau 3-folds). Calabi-Yau 4-folds have also recently appeared in connection with F-theory compactification to 4 dimensions [4, 5, 6, 7, 2]. The cases of  $Sp(2)$  holonomy manifolds (8 dimensional) and  $G_2$  and  $Spin(7)$  end up giving us compactifications below 4 dimensions.

### 1.2.1 Toroidal Compactifications

The space with maximal number of covariantly constant spinors is the flat torus  $T^d$ . This is also the easiest to describe the string propagation in. The main modification to the construction of the Hilbert space from flat non-compact space in this case involves relaxing the condition  $P_L = P_R$  because the string can wrap around the internal space and so  $X$  does not need to come back to itself as we go around  $\sigma$ . In particular if we consider compactification on a circle of radius  $R$  we can have

$$(P_L, P_R) = (\frac{n}{2R} + mR, \frac{n}{2R} - mR).$$

Here  $n$  labels the center of mass momentum of the string along the circle and  $m$  labels how many times the string is winding around the circle. Note that the spectrum of allowed  $(P_L, P_R)$  is invariant under  $R \rightarrow 1/2R$ . All that we have to do is to exchange the momentum and winding modes ( $n \leftrightarrow m$ ). This symmetry is a consequence of what is known as  $T$ -duality [8].

If we compactify on a  $d$ -dimensional torus  $T^d$  it can be shown that  $(P_L, P_R)$  belong to a  $2d$  dimensional lattice with signature  $(d, d)$ . Moreover this lattice is integral, self-dual and even. Evenness means,  $P_L^2 - P_R^2$  is even for each lattice vector. Self-duality means that any vector which has integral product with all the vectors in the lattice sits in the lattice as well. It is an easy exercise to check these condition in the one dimensional circle example given above. Note that we can change the radii of the torus and this will clearly affect the  $(P_L, P_R)$ . Given any choice of a  $d$ -dimensional torus compactifications, all the other ones can be obtained by doing an  $SO(d, d)$  Lorentz boost on  $(P_L, P_R)$  vectors. Of course rotating  $(P_L, P_R)$  by an  $O(d) \times O(d)$  transformation does not change the spectrum of the string states, so the totality of such vectors is given by

$$\frac{SO(d, d)}{SO(d) \times SO(d)}.$$

Some Lorentz boosts will not change the lattice and amount to relabeling the states. These are the boosts that sit in  $O(d, d; \mathbb{Z})$  (i.e. boosts with integer coefficients),



because they can be undone by choosing a new basis for the lattice by taking an integral linear combination of lattice vectors. So the space of inequivalent choices are actually given by

$$\frac{SO(d, d)}{SO(d) \times SO(d) \times O(d, d; Z)}.$$

The  $O(d, d; Z)$  generalizes the T-duality considered in the 1-dimensional case.

### 1.2.2 Compactifications on $K3$

The four dimensional manifold  $K3$  is the only compact four dimensional manifold, besides  $T^4$ , which admits covariantly constant spinors. In fact it has exactly half the number of covariantly constant spinors as on  $T^4$  and thus preserves half of the supersymmetry that would have been preserved upon toroidal compactification. More precisely the holonomy of a generic four manifold is  $SO(4)$ . If the holonomy resides in an  $SU(2)$  subgroup of  $SO(4)$  which leaves an  $SU(2)$  part of  $SO(4)$  untouched, we end up with one chirality of  $SO(4)$  spinor being unaffected by the curvature of  $K3$ , which allows us to define supersymmetry transformations as if  $K3$  were flat (note a spinor of  $SO(4)$  decomposes as  $(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})$  of  $SU(2) \times SU(2)$ ).

There are a number of realizations of  $K3$ , which are useful depending on which question one is interested in. Perhaps the simplest description of it is in terms of *orbifolds*. This description of  $K3$  is very close to toroidal compactification and differs from it by certain discrete isometries of the  $T^4$  which are used to (generically) identify points which are in the same *orbit* of the discrete group. Another description is as a 19 complex parameter family of  $K3$  defined by an algebraic equation.

Consider a  $T^4$  which for simplicity we take to be parametrized by four real coordinates  $x_i$  with  $i = 1, \dots, 4$ , subject to the identifications  $x_i \sim x_i + 1$ . It is sometimes convenient to think of this as two complex coordinates  $z_1 = x_1 + ix_2$  and  $z_2 = x_3 + ix_4$  with the obvious identifications. Now we identify the points on the torus which are mapped to each other under the  $Z_2$  action (involution) given by reflection in the coordinates  $x_i \rightarrow -x_i$ , which is equivalent to

$$z_i \rightarrow -z_i.$$

Note that this action has  $2^4 = 16$  fixed points given by the choice of midpoints or the origin in any of the four  $x_i$ . The resulting space is singular at any of these 16 fixed points because the angular degree of freedom around each of these points is cut by half. Put differently, if we consider any primitive loop going ‘around’ any of these 16 fixed point, it corresponds to an open curve on  $T^4$  which connects pairs of points related by the  $Z_2$  involution. Moreover the parallel transport of vectors along this path, after using the  $Z_2$  identification, results in a flip of the sign of the vector. This is true no matter how small the curve is. This shows that we cannot have a smooth manifold at the fixed points.

When we move away from the orbifold points of  $K3$  the description of the geometry of  $K3$  in terms of the properties of the  $T^4$  and the  $Z_2$  twist become less relevant, and it is natural to ask about other ways to think about  $K3$ . In general a simple way to define complex manifolds is by imposing complex equations in a compact space known as the projective  $n$ -space  $\mathbf{CP}^n$ . This is the space of complex variables  $(z_1, \dots, z_{n+1})$  excluding the origin and subject to the identification

$$(z_1, \dots, z_{n+1}) \sim \lambda(z_1, \dots, z_{n+1}) \quad \lambda \neq 0.$$

One then considers the vanishing locus of a homogeneous polynomial of degree  $d$ ,  $W_d(z_i) = 0$  to obtain an  $n - 1$  dimensional subspace of  $\mathbf{CP}^n$ . An interesting special case is when the degree is  $d = n + 1$ . In this case one obtains an  $n - 1$  complex dimensional manifold which admits a Ricci-flat metric. This is the case known as Calabi-Yau. For example, if we take the case  $n = 2$ , by considering cubics in it

$$z_1^3 + z_2^3 + z_3^3 + az_1z_2z_3 = 0$$

we obtain an elliptic curve, i.e. a torus of complex dimension 1 or real dimension 2. The next case would be  $n = 3$  in which case, if we consider a quartic polynomial in  $\mathbf{CP}^3$  we obtain the 2 complex dimensional  $K3$  manifold:

$$W = z_1^4 + z_2^4 + z_3^4 + z_4^4 + \text{deformations} = 0.$$

There are 19 inequivalent quartic terms we can add. This gives us a 19 dimensional complex subspace of 20 dimensional complex moduli of the  $K3$  manifold. Clearly

this way of representing  $K3$  makes the complex structure description of it very manifest, and makes the Kahler structure description implicit.

Note that for a generic quartic polynomial the  $K3$  we obtain is non-singular. This is in sharp contrast with the orbifold construction which led us to 16 singular points. It is possible to choose parameters of deformation which lead to singular points for  $K3$ . For example if we consider

$$z_1^4 + z_2^4 + z_3^4 + z_4^4 + 4z_1z_2z_3z_4 = 0$$

it is easy to see that the resulting  $K3$  will have a singularity (one simply looks for non-trivial solutions to  $dW = 0$ ).

There are other ways to construct Calabi-Yau manifolds and in particular  $K3$ 's. One natural generalization to the above construction is to consider weighted projective spaces where the  $z_i$  are identified under different rescalings. In this case one considers quasi-homogeneous polynomials to construct submanifolds.

### 1.2.3 Calabi-Yau Threefolds

Calabi-Yau threefolds are manifolds with  $SU(3)$  holonomy. The compactification on manifolds of  $SU(3)$  holonomy preserves 1/4 of the supersymmetry. In particular if we compactify  $N = 2$  theories on Calabi-Yau threefolds we obtain  $N = 2$  theories in  $d = 4$ , whereas if we consider  $N = 1$  theories we obtain  $N = 1$  theories in  $d = 4$ .

If we wish to construct the Calabi-Yau threefolds as toroidal orbifolds we need to consider six dimensional tori, three complex dimensional, which have discrete isometries residing in  $SU(3)$  subgroup of the  $O(6) = SU(4)$  holonomy group. A simple example is if we consider the product of three copies of  $T^2$  corresponding to the Hexagonal lattice and mod out by a simultaneous  $\mathbf{Z}_3$  rotation on each torus (this is known as the 'Z-orbifold'). This  $\mathbf{Z}_3$  transformation has 27 fixed points which can be blown up to give rise to a smooth Calabi-Yau.

We can also consider description of Calabi-Yau threefolds in algebraic geometry terms for which the complex deformations of the manifold can be typically realized

as changes of coefficients of defining equations, as in the  $K3$  case. For instance we can consider the projective 4-space  $\mathbf{CP}^4$  defined by 5 complex not all vanishing coordinates  $z_i$  up to overall rescaling, and consider the vanishing locus of a homogeneous degree 5 polynomial

$$P_5(z_1, \dots, z_5) = 0.$$

This defines a Calabi-Yau threefold, known as the quintic three-fold. This can be generalized to the case of product of several projective spaces with more equations. Or it can be generalized by taking the coordinates to have different homogeneity weights. This will give a huge number of Calabi-Yau manifolds.

### 1.3 Solitons in String Theory

Solitons arise in field theories when the vacuum configuration of the field has a non-trivial topology which allows non-trivial wrapping of the field configuration at spatial infinity around the vacuum manifold. These will carry certain topological charge related to the ‘winding’ of the field configuration around the vacuum configuration. Examples of solitons include magnetic monopoles in four dimensional non-abelian gauge theories with unbroken  $U(1)$ , cosmic strings and domain walls. The solitons naively play a less fundamental role than the fundamental fields which describe the quantum field theory. In some sense we can think of the solitons as ‘composites’ of more fundamental elementary excitations. However as is well known, at least in certain cases, this is just an illusion. In certain cases it turns out that we can reverse the role of what is fundamental and what is composite by considering a different regime of parameter. In such regimes the soliton may be viewed as the elementary excitation and the previously viewed elementary excitation can be viewed as a soliton. A well known example of this phenomenon happens in 2 dimensional field theories. Most notably the boson/fermion equivalence in the two dimensional sine-Gordon model, where the fermions may be viewed as solitons of the sine-Gordon model and the boson can be viewed as a composite of fermion-anti-fermion excitation. Another example is the T-duality we have already discussed in

the context of 2 dimensional world sheet of strings which exchanges the radius of the target space with its inverse. In this case the winding modes may be viewed as the solitons of the more elementary excitations corresponding to the momentum modes. As discussed before  $R \rightarrow 1/R$  exchanges momentum and winding modes. In anticipation of generalization of such dualities to string theory, it is thus important to study various types of solitons that may appear in string theory.

As already mentioned solitons typically carry some conserved topological charge. However in string theory every conserved charge is a gauge symmetry. In fact this is to be expected from a theory which includes quantum gravity. This is because the global charges of a black hole will have no influence on the outside and by the time the black hole disappears due to Hawking radiation, so does the global charges it may carry. So the process of formation and evaporation of black hole leads to a non-conservation of global charges. Thus for any soliton, its conserved topological charge must be a gauge charge. This may appear to be somewhat puzzling in view of the fact that solitons may be point-like as well as string-like, sheet-like etc. We can understand how to put a charge on a point-like object and gauge it. But how about the higher dimensional extended solitonic states? Note that if we view the higher dimensional solitons as made of point-like structures the soliton has no stability criterion as the charge can disintegrate into little bits.

Let us review how it works for point particles (or point solitons): We have a 1-form gauge potential  $A_\mu$  and the coupling of the particle to the gauge potential involves weighing the world-line propagating in the space-time with background  $A_\mu$  by

$$Z \rightarrow Z \exp(i \int_\gamma A),$$

where  $\gamma$  is the world line of the particle. The gauge principle follows from defining an action in terms of  $F = dA$ :

$$S = \int F \wedge *F, \tag{1.7}$$

where  $*F$  is the dual of the  $F$ , where we note that shifting  $A \rightarrow d\epsilon$  for arbitrary function  $\epsilon$  will not modify the action.

Suppose we now consider instead of a point particle a  $p$ -dimensional extended object. In this convention  $p = 0$  corresponds to the case of point particles and  $p = 1$  corresponds to strings and  $p = 2$  corresponds to membranes, etc. We shall refer to  $p$ -dimensional extended objects as  $p$ -branes (generalizing ‘membrane’). Note that the world-volume of a  $p$ -brane is a  $p + 1$  dimensional subspace  $\gamma_{p+1}$  of space-time. To generalize what we did for the case of point particles we introduce a gauge potential which is a  $p + 1$  form  $A_{p+1}$  and couple it to the charged  $p + 1$  dimensional state by

$$Z \rightarrow Z \exp(i \int_{\gamma_{p+1}} A_{p+1}).$$

Just as for the case of the point particles we introduce the field strength  $F = dA$  which is now a totally antisymmetric  $p+2$  tensor. Moreover we define the action as in (1.7), which possesses the gauge symmetry  $A \rightarrow d\epsilon$  where  $\epsilon$  is a totally antisymmetric tensor of rank  $p$ .

### 1.3.1 Magnetically Charged States

The above charge defines the generalization of electrical charges for extended objects. Can we generalize the notion of magnetic charge? Suppose we have an electrically charged particle in a theory with space-time dimension  $D$ . Then we measure the electrical charge by surrounding the point by an  $S^{D-2}$  sphere and integrating  $*F$  (which is a  $D - 2$  form) on it, i.e.

$$Q_E = \int_{S^{D-2}} *F.$$

Similarly it is natural to define the magnetic charge. In the case of  $D = 4$ , i.e. four dimensional space-time, the magnetically charged point particle can be surrounded also by a sphere and the magnetic charge is simply given by

$$Q_M = \int_{S^2} F.$$

Now let us generalize the notion of magnetic charged states for arbitrary dimensions  $D$  of space-time and arbitrary electrically charged  $p$ -branes. From the above description it is clear that the role that  $*F$  plays in measuring the electric charge

is played by  $F$  in measuring the magnetic charge. Note that for a  $p$ -brane  $F$  is  $p + 2$  dimensional, and  $*F$  is  $D - p - 2$  dimensional. Moreover, note that a sphere surrounding a  $p$ -brane is a sphere of dimension  $D - p - 2$ . Note also that for  $p = 0$  this is the usual situation. For higher  $p$ , a  $p$ -dimensional subspace of the space-time is occupied by the extended object and so the position of the object is denoted by a point in the transverse  $(D - 1) - p$  dimensional space which is surrounded by an  $S^{D-p-2}$  dimensional sphere.

Now for the magnetic states the role of  $F$  and  $*F$  are exchanged:

$$F \leftrightarrow *F.$$

To be perfectly democratic we can also define a magnetic gauge potential  $\tilde{A}$  with the property that

$$d\tilde{A} = *F = *dA.$$

In particular noting that  $F$  is a  $p + 2$  form, we learn that  $*F$  is an  $D - p - 2$  form and thus  $\tilde{A}$  is an  $D - p - 3$  form. We thus deduce that the magnetic state will be an  $D - p - 4$ -brane (i.e. one dimension lower than the degree of the magnetic gauge potential  $\tilde{A}$ ). Note that this means that if we have an electrically charged  $p$ -brane, with a magnetically charged dual  $q$ -brane then we have

$$p + q = D - 4. \tag{1.8}$$

This is an easy sum rule to remember. Note in particular that for a 4-dimensional space-time an electric point charge ( $p = 0$ ) will have a dual magnetic point charge ( $q = 0$ ). Moreover this is the only space-time dimension where both the electric and magnetic dual can be point-like.

Note that a  $p$ -brane wrapped around an  $r$ -dimensional compact object will appear as a  $p - r$ -brane for the non-compact space-time. This is in accord with the fact that if we decompose the  $p + 1$  gauge potential into an  $(p + 1 - r) + r$  form consisting of an  $r$ -form in the compact direction we will end up with an  $p + 1 - r$  form in the non-compact directions. Thus the resulting state is charged under the

left-over part of the gauge potential. A particular case of this is when  $r = p$  in which case we are wrapping a  $p$ -dimensional extended object about a  $p$ -dimensional closed cycle in the compact directions. This will leave us with point particles in the non-compact directions carrying ordinary electric charge under the reduced gauge potential which now is a 1-form.

### 1.3.2 String Solitons

From the above discussion it follows that the charged states will in principle exist if there are suitable gauge potentials given by  $p + 1$ -forms. Let us first consider type II strings. Recall that from the NS-NS sector we obtained an anti-symmetric 2-form  $B_{\mu\nu}$ . This suggests that there is a 1-dimensional extended object which couples to it by

$$\exp(i \int B).$$

But that is precisely how  $B$  couples to the world-sheet of the fundamental string. We thus conclude that *the fundamental string carries electric charge under the antisymmetric field  $B$* . What about the magnetic dual to the fundamental string? According to (1.8) and setting  $d = 10$  and  $p = 1$  we learn that the dual magnetic state will be a 5-brane. Note that as in the field theories, we expect that in the perturbative regime for the fundamental fields, the solitons be very massive. This is indeed the case and the 5-brane magnetic dual can be constructed as a solitonic state of type II strings with a mass per unit 5-volume going as  $1/g_s^2$  where  $g_s$  is the string coupling. Conversely, in the strong coupling regime these 5-branes are light and at infinite coupling they become massless, i.e. tensionless 5-branes [9].

Let us also recall that type II strings also have anti-symmetric fields coming from the R-R sector. In particular for type IIA strings we have 1-form  $A_\mu$  and 3-form  $C_{\mu\nu\rho}$  gauge potentials. Note that the corresponding magnetic dual gauge fields will be 7-forms and 5-forms respectively (which are not independent degrees of freedom). We can also include a 9-form potential which will have trivial dynamics in 10 dimensions. Thus it is natural to define a generalized gauge field  $\mathcal{A}$  by taking



the sum over all odd forms and consider the equation  $\mathcal{F} = *\mathcal{F}$  where  $\mathcal{F} = d\mathcal{A}$ . A similar statement applies to the type IIB strings where from the R-R sector we obtain all the even-degree gauge potentials (the case with degree zero can couple to a “-1-brane” which can be identified with an instanton, i.e. a point in space-time). We are thus led to look for p-branes with even  $p$  for type IIA and odd  $p$  for type IIB which carry charge under the corresponding RR gauge field. It turns out that surprisingly enough the states in the elementary excitations of string all are neutral under the RR fields. We are thus led to look for solitonic states which carry RR charge. Indeed there are such p-branes and they are known as  $D$ -branes [10, 11], as we will now review.

### 1.3.3 D-Branes

In the context of field theories constructing solitons is equivalent to solving classical field equations with appropriate boundary conditions. For string theory the condition that we have a classical solution is equivalent to the statement that propagation of strings in the corresponding background would still lead to a conformal theory on the worldsheet of strings, as is the case for free theories.

In search of such stringy p-branes, we are thus led to consider how could a p-brane modify the string propagation. Consider an  $p + 1$  dimensional plane, to be identified with the world-volume of the  $p$ -brane. Consider string propagating in this background. How could we modify the rules of closed string propagation given this  $p + 1$  dimensional sheet? The simplest way turns out to allow closed strings to open up and end on the  $p + 1$  dimensional world-volume. In other words we allow to have a new sector in the theory corresponding to open string with ends lying on this  $p + 1$  dimensional subspace. This will put Dirichlet boundary conditions on  $10 - p - 1$  coordinates of string endpoints. Such  $p$ -branes are called  $D$ -branes, with  $D$  reminding us of Dirichlet boundary conditions. In the context of type IIA,B we also have to specify what boundary conditions are satisfied by fermions. This turns out to lead to consistent boundary conditions only for  $p$  even for type IIA string and  $p$  odd for type IIB. This is a consequence of the fact that for type IIA(B), left-

right exchange is a symmetry only when accompanied by a  $Z_2$  spatial reflection with determinant  $-1(+1)$ . Moreover, it turns out that they do carry the corresponding RR charge [12].

Quantizing the new sector of type II strings in the presence of D-branes is rather straightforward. We simply consider the set of oscillators as before, but now remember that due to the Dirichlet boundary conditions on some of the components of string coordinates, the momentum of the open string lies on the  $p+1$  dimensional world-volume of the D-brane. It is thus straightforward to deduce that the massless excitations propagating on the D-brane will lead to the dimensional reduction of  $N = 1$ ,  $U(1)$  Yang-Mills from  $d = 10$  to  $p+1$  dimensions. In particular the  $10 - (p+1)$  scalar fields living on the D-brane, signify the D-brane excitations in the  $10 - (p+1)$  transverse dimensions. This tells us that the significance of the new open string subsector is to quantize the D-brane excitations.

An important property of D-branes is that when  $N$  of them coincide we get a  $U(N)$  gauge theory on their world-volume. This follows because we have  $N^2$  open string subsectors going from one D-brane to another and in the limit they are on top of each other all will have massless modes and we thus obtain the reduction of  $N = 1$   $U(N)$  Yang-Mills from  $d = 10$  to  $d = p+1$ .

Another important property of D-branes is that they are BPS states. A BPS state is a state which preserves a certain number of supersymmetries and as a consequence of which one can show that their mass (per unit volume) and charge are equal. This in particular guarantees their absolute stability against decay.

If we consider the tension of D-branes, it is proportional to  $1/g_s$ , where  $g_s$  is the string coupling constant. Note that as expected at weak coupling they have a huge tension. At strong coupling their tension goes to zero and they become tensionless.

We have already discussed that in  $K3$  compactification of string theory we end up with singular limits of manifolds when some cycles shrink to zero size. What is the physical interpretation of this singularity?

Suppose we consider for concreteness an  $n$ -dimensional sphere  $S^n$  with volume  $\epsilon \rightarrow 0$ . Then the string perturbation theory breaks down when  $\epsilon \ll g_s$ , where  $g_s$  is

the string coupling constant. If we have  $n$ -brane solitonic states such as D-branes then we can consider a particular solitonic state corresponding to wrapping the  $n$ -brane on the vanishing  $S^n$ . The mass of this state is proportional to  $\epsilon$ , which implies that in the limit  $\epsilon \rightarrow 0$  we obtain a massless soliton. An example of this is when we consider type IIA compactification on  $K3$  where we develop a singularity. Then by wrapping D2-branes around vanishing  $S^2$ 's of the singularity we obtain massless states, which are vectors. This in fact implies that in this limit we obtain enhanced gauge symmetry. Had we been considering type IIB on  $K3$  near the singularity, the lightest mode would be obtained by wrapping a D3-brane around vanishing  $S^2$ 's, which leaves us with a string state with tension of the order of  $\epsilon$  [13], [14]. This kind of regime which exists in other examples of compactifications as well is called the phase with tensionless strings [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34].

We could consider higher dimensional D-branes wrapping around the vanishing cycles and obtain tensionless  $p$ -branes with  $p > 1$ , but in such cases by dimensional analysis one can see that the relevant mass scale would be smallest for the smallest dimension D-brane.

## 1.4 From M-Theory to Tensionless Strings

M-theory is the hypothetical unification of several types of 10-dimensional strings [35, 36]. Its low-energy effective description is the 11-dimensional supergravity, and some valuable information can be extracted from its classical solutions. The basic dynamical objects of the M-theory are the 2-brane, which is electrically charged under the 3-form gauge field, and its magnetic dual, the 5-brane. The dynamics governing the M-brane interactions is by no means well-understood. Some of its features may be inferred, however, from compactification to string theory, where the RR charged  $p$ -branes have a remarkably simple description in terms of the D-branes [37, 12, 11].

The D-branes are objects on which the fundamental strings are allowed to end. There is evidence that a similar phenomenon takes place in M-theory: the funda-

mental 2-branes are allowed to have boundaries on the solitonic 5-branes [38, 15]. Thus, the 5-brane is the D-object of M-theory. The boundary of a 2-brane is a string, and the resulting boundary dynamics appears to reduce to a kind of string theory defined on the  $5 + 1$  dimensional world-volume. This picture has a number of interesting implications. Consider, for instance, two parallel 5-branes with a 2-brane stretched between them [15]. The two boundaries of the 2-brane give rise to two strings, lying within the first and second 5-branes respectively. The tension of these strings may be made arbitrarily small as the 5-branes are brought close together. In particular, it can be made much smaller than the Planck scale, which implies that the effective  $5 + 1$  dimensional string theory is decoupled from gravity [13]. While it is not clear how to describe such a string theory in world sheet terms, it has been suggested that its spectrum is given by the Green-Schwarz approach [39] ( $5 + 1$  is one of the dimensions where the Green-Schwarz string is classically consistent). In the limit of coincident 5-branes, we seem to find a theory of tensionless strings. These strings carry  $(0, 4)$  supersymmetry in  $5 + 1$  dimensions. Strings with  $(0, 2)$  supersymmetry were explored from several different points of view in refs. [17, 18, 19] for example.

## 2 APPEARANCE OF THE NULL BRANES

In the previous section we saw on examples how tensionless strings and branes may appear in modern string theory. Here we are going to consider these and other cases in some details.

### 2.1 Compactifications

In [15] it is shown that many of the  $p$ -branes of type II string theory and  $D = 11$  supergravity can have boundaries on other  $p$ -branes. The rules for when this can and cannot occur are derived from charge conservation. For example it is found that membranes in  $D = 11$  supergravity and IIA string theory can have boundaries on fivebranes. The boundary dynamics are governed by the self-dual  $D = 6$  string. A collection of  $N$  parallel fivebranes contains  $N(N - 1)/2$  self-dual strings which become tensionless as the fivebranes approach one another.

In [16] the author analyzes  $M$ -theory compactified on  $(K3 \times S^1)/Z_2$  where the  $Z_2$  changes the sign of the three form gauge field, acts on  $S^1$  as a parity transformation and on  $K3$  as an involution with eight fixed points preserving  $SU(2)$  holonomy. At a generic point in the moduli space the resulting theory has as its low energy limit  $N = 1$  supergravity theory in six dimensions with eight vector, nine tensor and twenty hypermultiplets. The gauge symmetry can be enhanced (*e.g.* to  $E_8$ ) at special points in the moduli space. At other special points in the moduli space tensionless strings appear in the theory.

In [17] T-duality is used to extract information on an instanton of zero size in the  $E_8 \times E_8$  heterotic string. The authors discuss the possibility of the appearance of a tensionless anti-self-dual non-critical string through an implementation of the mechanism suggested by Strominger of two coincident 5-branes [15]. It is argued that when an instanton shrinks to zero size a tensionless non-critical string appears at the core of the instanton. It is further conjectured that appearance of tensionless strings in the spectrum leads to new phase transitions in six dimensions in much the same way as massless particles do in four dimensions.

The paper [18] discusses the singularities in the moduli space of string compactifications to six dimensions with  $N = 1$  supersymmetry. Such singularities arise from either massless particles or non-critical tensionless strings. The points with tensionless strings are sometimes phase transition points between different phases of the theory. These results appear to connect all known  $N = 1$  supersymmetric six-dimensional vacua.

Heterotic strings on  $R^6 \times K3$  generically appear to undergo some interesting new phase transition at that value of the string coupling for which the one of the six-dimensional gauge field kinetic energies changes sign. An exception is the  $E_8 \times E_8$  string with equal instanton numbers in the two  $E_8$ 's, which admits a heterotic/heterotic self-duality. In [19] the dyonic string solution of the six-dimensional heterotic string is generalized to include non-trivial gauge field configurations corresponding to self-dual Yang-Mills instantons in the four transverse dimensions. It is found that vacua which undergo a phase transition always admit a string solution exhibiting a naked singularity, whereas for vacua admitting a self-duality the solution is always regular. When there is a phase transition, there exists a choice of instanton numbers for which the dyonic string is tensionless and quasi-anti-self-dual at that critical value of the coupling. For an infinite subset of the other choices of instanton number, the string will also be tensionless, but all at larger values of the coupling.

Phase transitions in  $M$ -theory and  $F$ -theory are studied in [20]. In  $M$ -theory compactification to five dimensions on a Calabi-Yau, there are topology-changing transitions similar to those seen in conformal field theory, but the non-geometrical phases known in conformal field theory are absent. At boundaries of moduli space where such phases might have been expected, the moduli space ends, by a conventional or unconventional physical mechanism. The unconventional mechanisms, which roughly involve the appearance of tensionless strings, can sometimes be better understood in  $F$ -theory.

When  $N$  five-branes of  $M$ -theory coincide, the world-volume theory contains tensionless strings, according to Strominger's construction. This suggests a large  $N$

limit of tensionless string theories. For the small  $E_8$  instanton theories, the definition would be a large instanton number. An adiabatic argument suggests [21] that in the large  $N$  limit an effective extra uncompactified dimension might be observed. In [21] a kind of “surface-equations” are also proposed and might describe correlators in the tensionless string theories. In these equations, the anti-self-dual two forms of 6D and the tensionless strings enter on an equal footing.

In [22] the authors argue for the existence of phase transitions in  $3+1$  dimensions associated with the appearance of tensionless strings. The massless spectrum of this theory does not contain a graviton: it consists of one  $N = 2$  vector multiplet and one linear multiplet, in agreement with the light-cone analysis of the Green-Schwarz string in  $3 + 1$  dimensions. In M-theory the string decoupled from gravity arises when two 5-branes intersect over a three-dimensional hyperplane. The two 5-branes may be connected by a 2-brane, whose boundary becomes a tensionless string with  $N = 2$  supersymmetry in  $3 + 1$  dimensions.

A class of four dimensional  $N=1$  compactifications of the  $SO(32)$  heterotic/type I string theory which are destabilized by nonperturbatively generated superpotentials are described in [23]. In the type I description, the destabilizing superpotential is generated by a one instanton effect or gaugino condensation in a nonperturbative  $SU(2)$  gauge group. The dual, heterotic description involves destabilization due to world-sheet instanton or *half* world-sheet instanton effects in the two cases. The analysis performed, also suggests that the tensionless strings which arise in the  $E_8 \times E_8$  theory in six dimensions when an instanton shrinks to zero size should, in some cases, have supersymmetry breaking dynamics upon further compactification to four dimensions.

In the article [24] the appearance of tensionless strings in M-theory is examined. These tensionless strings are subsequently interpreted in a string theory context. In particular, tensionless strings appearing in M-theory on  $S^1$ , M-theory on  $S^1/\mathbf{Z}_2$ , and M-theory on  $T^2$  are examined. An interpretation is given for the appearance of such strings in a string theory context. Then the appearance of some tensionless strings in string theory is examined. Subsequently the author interprets these tensionless

strings in a M-theory context.

In [25] F-theory [4] on elliptic threefold Calabi-Yau near colliding singularities is studied. It is demonstrated that resolutions of those singularities generically correspond to transitions to phases characterized by new tensor multiplets and enhanced gauge symmetry. These are governed by the dynamics of tensionless strings.

The work [26] is devoted to the examination of different aspects of Calabi-Yau four-folds as compactification manifolds of F-theory, using mirror symmetry of toric hypersurfaces. A discussion is given on the physical properties of the space-time theories, for a number of examples which are dual to  $E_8 \times E_8$  heterotic  $N = 1$  theories. Non-critical strings of various kinds, with low tension for special values of the moduli, lead to interesting physical effects. A complete classification is given of those divisors in toric manifolds that contribute to the non-perturbative four-dimensional superpotential; the physical singularities associated to it are related to the appearance of tensionless strings. In some cases non-perturbative effects generate an everywhere non-zero quantum tension leading to a combination of a conventional field theory with light strings hiding at a low energy scale related to supersymmetry breaking.

Type IIB strings compactified on K3 have a rich structure of solitonic strings, transforming under  $SO(21, 5, Z)$ . In [14] the BPS tension formula for these strings is derived, and their properties, in particular, the points in the moduli space where certain strings become tensionless are discussed. By examining these tensionless string limits, the authors shed some further light on the conjectured dual M-theory description of this compactification.

In [27] the authors study critical points of the BPS mass  $Z$ , the BPS string tension  $Z_m$ , the black hole potential  $V$  and the gauged central charge potential  $P$  for M-theory compactified on Calabi-Yau three-folds. They first show that the stabilization equations for  $Z$  (determining the black hole entropy) take an extremely simple form in five dimensions as opposed to four dimensions. The stabilization equations for  $Z_m$  are also very simple and determine the size of the infinite  $AdS_3$ -throat of the string. The black hole potential in general exhibits two classes of critical points:



supersymmetric critical points which coincide with those of the central charge and non-supersymmetric critical points. Then the discussion is generalized to the entire extended Kähler cone encompassing topologically different but birationally equivalent Calabi-Yau three-folds that are connected via flop transitions. The behavior of the four potentials is examined to probe the nature of these phase transitions. It is found that  $V$  and  $P$  are continuous but not smooth across the flop transition, while  $Z$  and its first two derivatives, as well as  $Z_m$  and its first derivative, are continuous. This in turn implies that supersymmetric stabilization of  $Z$  and  $Z_m$  for a given configuration takes place in at most one point throughout the entire extended Kähler cone. The corresponding black holes (or string states) interpolate between different Calabi-Yau three-folds. At the boundaries of the extended Kähler cone electric states become massless and/or magnetic strings become tensionless.

In [28] it is shown how Higgs mechanism for non-abelian  $N = 2$  gauge theories in four dimensions is geometrically realized in the context of type II strings as transitions among compactifications of Calabi-Yau threefolds. This result and T-duality are used for a further compactification on a circle to derive  $N = 4$ ,  $D = 3$  dual field theories. This reduces dualities for  $N = 4$  gauge systems in three dimensions to perturbative symmetries of string theory. Moreover, the dual of a gauge system always exists but may or may not correspond to a lagrangian system. In particular the conjecture of Intriligator and Seiberg is verified that an ordinary gauge system is dual to compactification of exceptional tensionless string theory down to three dimensions.

In [29] using geometric engineering in the context of type II strings, the authors obtain exact solutions for the moduli space of the Coulomb branch of all  $N = 2$  gauge theories in four dimensions involving products of  $SU$  gauge groups with arbitrary number of bi-fundamental matter for chosen pairs, as well as an arbitrary number of fundamental matter for each factor. Asymptotic freedom restricts the possibilities to  $SU$  groups with bi-fundamental matter chosen according to ADE or affine ADE Dynkin diagrams. It is found that in certain cases the solution of the Coulomb branch for  $N = 2$  gauge theories is given in terms of a three dimensional complex

manifold rather than a Riemann surface. A new stringy strong coupling fixed points are studied, arising from the compactification of higher dimensional theories with tensionless strings. Applications are considered to three dimensional  $N = 4$  theories.

In [30] an analysis is made of the world-volume theory of multiple Kaluza-Klein monopoles in string and  $M$ - theory by identifying the appropriate zero modes of various fields. The results are consistent with supersymmetry, and all conjectured duality symmetries. In particular for  $M$ -theory and type IIA string theory, the low energy dynamics of  $N$  Kaluza-Klein monopoles is described by supersymmetric  $U(N)$  gauge theory, and for type IIB string theory, the low energy dynamics is described by a  $(2,0)$  supersymmetric field theory in  $(5+1)$  dimensions with  $N$  tensor multiplets and tensionless self-dual strings.

In [31] it is shown that six-dimensional supergravity coupled to tensor and Yang-Mills multiplets admits not one but two different theories as global limits, one of which was previously thought not to arise as a global limit and the other of which is new. The new theory has the virtue that it admits a global anti-self-dual string solution obtained as the limit of the curved-space gauge dyonic string, and can, in particular, describe tensionless strings. It is speculated that this global model can also represent the world-volume theory of coincident branes.

Certain properties of six-dimensional tensionless E-strings (arising from zero size  $E_8$  instantons) are studied in [32]. In particular, it is shown that  $n$  E-strings form a bound string which carries an  $E_8$  level  $n$  current algebra as well as a left-over conformal system with  $c = 12n - 4 - 248n/(n + 30)$ , whose characters can be computed. Moreover, it is shown that the characters of the  $n$ -string bound state are captured by  $N = 4$   $U(n)$  topological Yang-Mills theory on  $K3/2$ .

Novel 3+1 dimensional  $N = 2$  superconformal field theories with tensionless BPS string solitons are believed to arise when two sets of M5 branes intersect over a 3+1 dimensional hyperplane. A DLCQ (discrete light-cone quantization) description [2] of these theories is derived in [33] as supersymmetric quantum mechanics on the Higgs branch of suitable four dimensional  $N = 1$  supersymmetric gauge theories. This formulation allows one to determine the scaling dimensions of certain chiral

primary operators in the conformal field theories.

In [34]  $N = 2$  superconformal theories defined on a  $3 + 1$  dimensional hyperplane intersection of two sets of M5 branes are considered. These theories have tensionless BPS string solitons. A dual supergravity formulation is used to deduce some of their properties via the AdS/CFT correspondence [40, 41].

In [42]  $N = 1$  four dimensional gauge theories are studied as the world-volume theory of D4-branes between NS 5-branes. A mechanism is proposed for enhanced chiral symmetry in the brane construction which is associated with tensionless *three-branes* in six dimensions. This mechanism can be explained as follows. As is, by now well known, quantization of open strings lead to massless hypermultiplets whenever two D-branes (which break to  $1/4$  of the supersymmetry) meet in space. It is not known however what are the states which get massless when a D brane meets a NS brane. The analysis performed in [42] predicts that the states are vector multiplets. The only virtual states which end on both a NS brane and a D6 brane are D4 branes. There is no other brane which has this property. Thus we can have virtual open D4-branes which have three-brane boundaries which propagate on the world-volume of the D6 and NS branes. When these two branes touch, the tension of the three-branes vanishes. The world-volume of the D4-branes consists of 0123 and a real line in the 456 space which connects the NS and D6 branes. A supersymmetric configuration which is consistent with the supersymmetries in this problem implies that the D6 and NS branes will have identical 45 positions and different  $x^6$  positions. This gives a special case to the point where in the field theory the masses are zero and chiral symmetry is expected. Thus one may predict that quantization of tensionless three-branes in six dimensions gives rise to massless vector multiplet in six dimensions.

A simple proof is given in [43] of the known S-duality of heterotic string theory compactified on a  $T^6$ . Using this S-duality the tensions for a class of BPS 5-branes in heterotic string theory on a  $S^1$  is calculated. One of these, the Kaluza-Klein monopole, becomes tensionless when the radius of the  $S^1$  is equal to the string length. Then the question of stability of the heterotic NS 5-brane with a transverse circle is studied. It turns out that for large radii the NS 5-brane is absolutely stable.

## 2.2 High Energy Limit

The characteristic scale of string theory is given by the string tension  $T_1 = (2\pi\alpha')^{-1}$ . At energies of the order of  $\sqrt{T_1}$  or higher, string physics truly distinguishes itself from point particle physics and various high energy limits have been studied to gain insight into the elusive physical basis of the theory. One may view the high energy limit of string theory as a *zero tension limit*, since only the energy measured in string units,  $E/\sqrt{T_1}$ , is relevant. There are different ways of taking the limit  $T_1 \mapsto 0$  in the full theory. A natural choice is to take  $T_1 \mapsto 0$  in the string action. One may then ask if the resulting null string can be quantized and if interactions can be introduced. All this applies also to the higher dimensional string generalizations - the null  $p$ -branes of different kind.

Another reason for studying the zero tension limit of  $p$ -branes is the duality between strings and 5-branes [9]. As we have already discussed in the Introduction, the string theory admits 5-branes as solitonic solution, and vice versa, the 5-brane theory admits strings as solitonic solutions. Moreover, the 5-brane tension  $T_5$  and the string tension  $T_1$  are related through a Dirac type quantization rule

$$\kappa^2 T_1 T_5 = n\pi,$$

where  $\kappa$  is the gravitational constant and  $n$  is an integer. We see from this relation that the small tension region of 5-branes is related to the large tension region of strings, so studying the zero tension limit of  $p$ -branes could actually teach us more about strings.

A further reason why it could be interesting to study such tensionless extended objects is the appearance of null  $Dp$ -branes in the high energy limit. As is known, static and moving  $Dp$ -branes can either be perceived as space-like hyper-surfaces on which open strings can end, or they can equivalently be described by  $Dp$ -brane boundary states into which closed strings can disappear.  $Dp$ -branes are characterized by a tension  $T_{Dp}$  ( $\neq 0$ ) which can be computed by considering the exchange of closed strings between two  $Dp$ -branes [12]. Because of this non-vanishing tension these branes are either static, or they move with velocities which are less than the

speed of light. A natural problem to pose is whether it is possible to extend the notion of  $Dp$ -branes above to also include  $Dp$ -branes which move at the speed of light, or equivalently  $Dp$ -branes which have a vanishing tension  $T_{Dp} = 0$ , i.e. tensionless  $Dp$ -branes.

The expression for  $T_{Dp}$  in terms of the inverse of the fundamental string tension  $2\pi\alpha'$ , and the string coupling  $g_s$ , is given by

$$T_{Dp} = (2\pi)^{-(p-1)/2} g_s^{-1} (2\pi\alpha')^{-(p+1)/2} \equiv (2\pi)^{-(p-1)/2} g_s^{-1} T_p, \quad (2.9)$$

where  $p \geq 0$  is the number of space-like directions in the brane. When this expression is extrapolated to arbitrarily large values of  $g_s$ ,  $T_{Dp}$  can attain values which are arbitrarily close to zero. Hence, in the strongly coupled regime  $Dp$ -branes become light states (since their masses  $M_{Dp}$  behaves as  $M_{Dp} \sim g_s^{-1}$ ), and when  $g_s = \infty$  the  $Dp$ -branes will become tensionless. This behaviour lies at the very foundation of M(atrix)-theory [44].

## 2.3 Gravity and Cosmology

The investigations in the domain of classical and quantum string propagation in curved space-times are relevant for the physics of quantum gravitation as well as for the understanding of the cosmic string models in cosmology [45].

As we have already mention, strings are characterized by an energy scale  $\sqrt{T_1}$ . The frequencies of the string modes are proportional to  $T_1$  and the length of the string scales with  $1/\sqrt{T_1}$ . The gravitational field provides another length scale, the curvature radius of the space-time  $R_c$ . For a string moving in a gravitational field a useful parameter is the dimensionless constant  $C = R_c \sqrt{T_1}$ . Large values of  $C$  imply weak gravitational field ( the metric does not change appreciably over distances of the order of the string length). We may reach large values of  $C$  by letting  $T_1 \rightarrow \infty$ . In this limit the string shrinks to a point. In the opposite limit, small values of  $C$ , we encounter strong gravitational fields and it is appropriate to consider  $T_1 \rightarrow 0$ , i.e. null or tensionless strings.

It turns out that the concept of null strings also appear in the strong coupling

limit of *two dimensional* gravity [46]. The gravitational constant  $\kappa$  is the analog of the Regge slope ( $\alpha'$ ) and when  $\kappa \rightarrow \infty$ , 2 dimensional quantum gravity can be understood as a tensionless string theory embedded in a two-dimensional target space. The temporal coordinate of the target space play the role of the time and the wave function can be interpreted as in standard quantum mechanics.

### 2.3.1 The Null String Approach

The string equations of motion and constraints in curved space-time are highly nonlinear and, in general, not exactly solvable. There are different methods available to solve the string equations of motion and constraints in curved space-times (for a review see for example [47]). These are the string perturbation approach, the null string approach, the  $\tau$ -expansion, and the construction of global solutions (for instance by solitonic and inverse scattering methods). A general approximation method is the null string approach [48]. In such approach the string equations of motion and constraints are systematically expanded in powers of  $c$  (the speed of light in the world-sheet). This corresponds to a small string tension expansion. At zeroth order, the string is effectively equivalent to a continuous beam of massless particles labeled by the world-sheet spatial parameter  $\sigma$ . The points on the string do not interact between them but they interact with the gravitational background.

In [49] quantum bosonic strings in strong gravitational fields are studied. Within the systematic expansion introduced in [48] one obtains to zeroth order the null string, while the first order correction incorporates the string dynamics. This formalism is applied to quantum null strings in de Sitter space-time. After a reparametrization of the world-sheet coordinates, the equations of motion are simplified. The quantum algebra generated by the constraints is considered, ordering the momentum operators to the right of the coordinate operators. No critical dimension appears. It is anticipated however that the conformal anomaly will appear when the first order corrections proportional to string tension  $T_1$  are introduced.

The classical dynamics of a bosonic string in the  $D$ -dimensional Friedmann–Robertson–Walker (with  $k = 0$ ) and Schwarzschild backgrounds is investigated in

[50]. This is done by making a perturbative development in the string coordinates around a null string configuration. The background geometry is taken into account exactly.

The dynamics of a string near a Kaluza-Klein black hole is studied in [51]. Solutions to the classical string equations of motion are obtained using the world sheet velocity of light as an expansion parameter, i.e. the null string expansion. The electrically and magnetically charged cases are considered separately. Solutions for string coordinates are obtained in terms of the world-sheet coordinate  $\tau$ . It is shown that the Kaluza-Klein radius increases/decreases with  $\tau$  for electrically/magnetically charged black hole.

The tension as a perturbative parameter in the nonlinear string equations of motion in curved space-times is also considered in [52, 53, 54, 55].

### 2.3.2 Null Branes in Cosmology

In general, in any cosmological model based on string theory, one has to face a regime of strong coupling and large curvatures when approaching the big-bang singularity. In the strong coupling regime, D-branes are the fundamental players. D-branes may give new insight into the understanding of the cosmological evolution of the Universe at early epochs. The authors of [56] analyze the dynamics of D-branes in curved backgrounds and discuss the parameter space of M-theory as a function of the coupling constant and of the curvature of the Universe. They show that D-branes may be efficiently produced by gravitational effects. Furthermore, in curved spacetimes the transverse fluctuations of the D-branes develop a tachyonic mode and when the fluctuations grow larger than the horizon, the branes become tensionless and break up. This signals a transition to a new regime. They also comment on possible implications for the so-called *brane world* scenario, where the Standard Model gauge and matter fields live inside some branes while gravitons live in the bulk.

The paper [57] considers the dynamics of null bosonic  $p$ -branes in curved spaces. It is shown that their motion equations can be linearized and exactly solved (in the

case of Robertson-Walker space-time with  $k = 0$ , i.e. with flat space-like section) in contrast to the case of tensile  $p$ -branes. It is found that the perfect fluid of null  $p$ -branes is an alternative dominant source of gravity in the Hilbert-Einstein equations for  $D$ -dimensional Friedmann universe with flat space-like section. This work is a generalization for  $p > 1$  of the results obtained earlier in [58].



### 3 CLASSICAL AND QUANTUM PROPERTIES OF THE NULL BRANES

Until now we learn how null branes of different dimension may appear in the context of contemporary string theory and in the connected with it theory of gravity and cosmology. At the same time, this explains why we are interested in more careful investigation of the classical and quantum properties of such extended relativistic objects.

In this section we begin the description of different models for tensionless  $p$ -branes. The particular case  $p = 1$ , i.e. the tensionless *strings*, which is much more studied by now, will not be considered here. We will concentrate our attention on the known results about the general case.

A Lagrangian which could describe under certain conditions null bosonic branes in  $D$ -dimensional Minkowski space-time was first proposed in [59]. An action for a tensionless  $p$ -brane with space-time supersymmetry was first given in [60, 61]. Since then, other types of actions and Hamiltonians (with and without supersymmetry) have been introduced and studied in the literature [62, 63, 64, 65, 66, 67, 68, 69]. Owing to their zero tension, the world-volume of the null  $p$ -branes is a light-like,  $(p + 1)$ -dimensional hypersurface, imbedded in the Minkowski space-time. Correspondingly, the determinant of the induced metric is zero. As in the tensile case, the null brane actions can be written in reparametrization and space-time conformally invariant form. However, their distinguishing feature is that at the classical level they may have any number of global space-time supersymmetries and be  $\kappa$ -invariant in all dimensions, which support Majorana (or Weyl) spinors. At the quantum level, they are anomaly free and do not exhibit any critical dimension, when appropriately chosen operator ordering is applied [70, 62, 63, 66, 71]. The only exception are the tensionless branes with manifest conformal invariance, with critical dimension  $D = 2$  for the bosonic case and  $D = 2 - 2N$  for the spinning case,  $N$  being the number of world-volume supersymmetries [66].

Let us mention also the paper [72], which is devoted to the construction of field

theory propagators of null strings and  $p$ -branes, as well as the corresponding spinning versions.

Almost all of the above investigations deal with *free* null branes moving in *flat* background (a qualitative consideration of null  $p$ -brane interacting with a scalar field has been done in [63]). The interaction of tensionless membranes ( $p = 2$ ) with antisymmetric background tensor field in four dimensional Minkowski space, described by means of Wess-Zumino-like action, is studied in [73]. The resulting equations of motion are integrated exactly.

To our knowledge, the only papers till now devoted to the classical dynamics of *null*  $p$ -branes ( $p \geq 2$ ) moving in *curved* space-times are [57, 74, 75, 76].

In [57], the null  $p$ -branes living in  $D$ -dimensional Friedmann-Robertson-Walker space-time with flat space-like section ( $k = 0$ ) have been investigated. The corresponding equations of motion have been solved exactly. It was argued that an ideal fluid of null  $p$ -branes may be considered as a source of gravity for Friedmann-Robertson-Walker universes.

In [74], the classical mechanics of the null branes in a gravity background was formulated. The Batalin-Fradkin-Vilkovisky (BFV) approach in its Hamiltonian version was applied to the considered dynamical system. Some exact solutions of the equations of motion and of the constraints for the null membrane in general stationary axially symmetric four dimensional gravity background were found. The examples of Minkowski, de Sitter, Schwarzschild, Taub-NUT and Kerr space-times were considered. Another exact solution, for the Demianski-Newman background, can be found in [75].

The article [76] considers null bosonic  $p$ -branes moving in curved space-times and a method for solving their equations of motion and constraints, which is suitable for string theory backgrounds is developed. As an application, an explicit exact solution for the ten dimensional solitonic five-brane gravity background is given.

Now, we are going to describe in more detail the results of works on null  $p$ -branes previous to ours.

To begin with, let us write down the Lagrangian density for the bosonic  $p$ -brane

given in [59]:

$$L = S_{\mu_0\mu_1\dots\mu_p}(\xi)\Sigma^{\mu_0\mu_1\dots\mu_p}(\xi), \quad \xi = (\xi^0, \xi^1, \dots, \xi^p). \quad (3.10)$$

Here

$$\Sigma^{\mu_0\mu_1\dots\mu_p}(\xi) = \epsilon^{J_0J_1\dots J_p}\partial_{J_0}x^{\mu_0}\partial_{J_1}x^{\mu_1}\dots\partial_{J_p}x^{\mu_p},$$

$\epsilon$  being the  $p$ -dimensional Levi-Civita symbol. In (3.10), the totally antisymmetric tensor  $S_{\mu_0\mu_1\dots\mu_p}(\xi)$  is restricted to lie on a single orbit of  $SO(1, d)_0$  in the space of  $p+1$  dimensional antisymmetric tensors. For suitable choices of the orbits for  $S(\xi)$ , one expects (3.10) to describe physically sensible  $p$ -dimensional objects including null ones.

In [60, 61], an action (3.18) for a space-time supersymmetric null  $p$ -brane is proposed in the following form

$$S_p = C \int d^{p+1}\xi \det(w_J^\mu w_K^\nu \eta_{\mu\nu}) / 2E, \quad (3.11)$$

$$w_J^\mu = \partial_J x^\mu - i\bar{\theta}\gamma^\mu \partial_J \theta, \quad (J, K = 0, 1, \dots, p),$$

where  $E(\xi)$  is a Lagrange multiplier. It is argued there that this action should possess Siegel  $\kappa$ -invariance [77] in *any* space-time dimension and for *any* number of supersymmetries, contrary to the tensionful superstrings and super  $p$ -branes.

In the paper [72], field theory propagators for null strings and  $p$ -branes, as well as for their generalizations to the spinning case are constructed by analogy with the massless relativistic particles. The propagator for the bosonic null  $p$ -brane is given by

$$G_b(x_2(\underline{\sigma}), x_1(\underline{\sigma})) = \int \mathcal{D}\nu(\underline{\sigma}) \nu(\underline{\sigma})^{-D/2} \exp\left(-\int d^p\sigma \frac{(\Delta x)^2}{2\nu(\underline{\sigma})}\right),$$

$$x^\mu(\tau, \underline{\sigma}) = x^\mu(\tau_1) + \frac{\Delta x^\mu}{\Delta\tau}(\tau - \tau_1) + Y^\mu(\tau, \underline{\sigma}), \quad \underline{\sigma} = (\sigma^1, \dots, \sigma^p)$$

where  $x_1$  and  $x_2$  are the initial and final brane configurations. The corresponding propagator for the spinning null  $p$ -brane is obtained to be

$$G_s(x_2(\underline{\sigma}), x_1(\underline{\sigma})) = \int \mathcal{D}\nu(\underline{\sigma}) \nu(\underline{\sigma})^{-1-D/2} \gamma^\mu \Delta x_\mu(\underline{\sigma}) \exp\left(-\int d^p\sigma \frac{(\Delta x)^2}{2\nu(\underline{\sigma})}\right),$$

where  $\gamma^\mu$  are the  $D$ -dimensional Dirac matrices, realizing the zero modes of the fermionic constraints.  $G_b$  and  $G_s$  satisfy the equation

$$\partial^2 G = \delta(x_2 - x_1),$$

$\partial^2$  being the  $(p+1)$ -dimensional d'Alembertian, as can be easily checked explicitly.

Let us now describe the main results obtained in [62]. These are:

1. It is proven that the null spinning  $p$ -brane is a system of rank 1.
2. An explicit expression for the propagator in the momentum space is found.
3. The question is discussed about how the critical dimensions are modified if the boundary conditions are changed.
4. It is argued that the functional diffusion equation is not modified when fermionic corrections are considered.

Now we are going to briefly explain how these results are obtained. The hamiltonian constraints for the spinning  $p$ -brane can be written as

$$T_0 = \eta^{\mu\nu} p_\mu p_\nu = 0, \quad T_j = p_\nu \partial_j x^\nu + \frac{i}{2} \sum_{J=0}^p \Gamma_J^\mu \partial_j \Gamma_{J\mu} = 0, \quad S_J = \Gamma_J^\mu p_\mu = 0, \\ J = (0, j).$$

Here  $\Gamma_J^\mu(\xi)$  are real variables and transform as spinors in the world-volume and as vectors in space-time. It is also supposed that they satisfy the Clifford algebra

$$\{\Gamma_J^\mu(\underline{\sigma}_1), \Gamma_K^\nu(\underline{\sigma}_2)\} = i\delta_{JK}\eta^{\mu\nu}\delta^p(\underline{\sigma}_1 - \underline{\sigma}_2).$$

As a consequence, the corresponding canonical Hamiltonian turns out to be

$$H_c = \int d^p\sigma (N^J T_J + i\bar{\lambda}^J S_J) = 0,$$

where  $N^J$  and  $\bar{\lambda}^J$  are Lagrange multipliers.

The constraints  $T_J$ ,  $S_J$  are first class and satisfy the algebra

$$\begin{aligned} [T_0(\underline{\sigma}_1), T_j(\underline{\sigma}_2)] &= -[T_0(\underline{\sigma}_1) + T_0(\underline{\sigma}_2)]\partial_j\delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\ [T_j(\underline{\sigma}_1), T_k(\underline{\sigma}_2)] &= [\delta_j^l T_k(\underline{\sigma}_1) + \delta_k^l T_j(\underline{\sigma}_2)]\partial_l\delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\ [T_0(\underline{\sigma}_1), T_0(\underline{\sigma}_2)] &= 0, \end{aligned}$$

$$\begin{aligned}
[S_J(\underline{\sigma}_1), T_0(\underline{\sigma}_2)] &= 0, \\
[S_J(\underline{\sigma}_1), T_k(\underline{\sigma}_2)] &= \frac{1}{2} [S_J(\underline{\sigma}_1) + 2S_J(\underline{\sigma}_2)] \partial_k \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\
\{S_J(\underline{\sigma}_1), S_K(\underline{\sigma}_2)\} &= 2i\delta_{JK} T_0(\underline{\sigma}_1) \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2).
\end{aligned}$$

Then the proper time gauge [78] is chosen

$$\dot{N}_0 = 0, \quad N_j = 0, \quad \dot{\lambda}_0 = 0, \quad \lambda_j = 0$$

and the author proceed with applying the BFV formalism [79, 80, 81, 82]. Correspondingly, the following extended phase space is introduced

$$(p_\mu, x^\mu, \Gamma_J^\mu) \oplus (\Pi_J, N^J, \rho_J, \lambda^J) \oplus (\eta_J, \bar{\mathcal{P}}^J, \bar{\eta}_J, \mathcal{P}^J, C_J, \bar{b}^J, \bar{C}_J, b^J),$$

where  $(\Pi_J, \rho_J)$  are the canonical momenta associated with the Lagrange multipliers  $(N^J, \lambda^J)$  while the dynamics remains unchanged provided we impose  $\Pi_J$  and  $\rho_J$  as new constraints, i.e.

$$\Pi_J = 0, \quad \rho_J = 0.$$

The variables  $(\eta_J, \bar{\eta}_J)$  and  $(C_J, \bar{C}_J)$  represent anticommuting and commuting ghosts respectively, with  $(\bar{\mathcal{P}}^J, \mathcal{P}^J, \bar{b}^J, b^J)$  as their conjugated momenta.

Using the constraint algebra, the Becchi-Rouet-Stora-Tyutin (BRST) charge is obtained to be

$$\begin{aligned}
\Omega &= \int d^p \sigma [\eta^J T_J + C^J S_J + \mathcal{P}^J \Pi_J + b^J \rho_J + \bar{\mathcal{P}}_0 \left( \partial_j \eta^0 \eta^j + \frac{1}{2} \eta^0 \partial_j \eta^j \right) \\
&+ \bar{\mathcal{P}}_j \left( \partial_k \eta^j \eta^k + \partial_k \eta^k \eta^j + \eta^k \partial_k \eta^j \right) - \bar{\mathcal{P}}_0 \left( \partial_j \eta^j C^0 + \frac{1}{2} \eta^j \partial_j C^0 - i C^0 C^0 - i C_j C^j \right) \\
&- \bar{b}_0 \left( \partial_j \eta^j C^0 + \frac{1}{2} \eta^j \partial_j C^0 \right) - \bar{b}_k \left( \partial_j \eta^j C^k - \frac{1}{2} \partial_k C^j \eta^k \right) + \bar{b}_0 \left( \partial_j C^0 \eta^j + \frac{1}{2} C^0 \partial_j \eta^j \right) \\
&- \frac{1}{2} \bar{b}_k \eta^j \partial_j C^k + \bar{b}_j \partial_k \eta^k \eta^j + \bar{b}_j C^k \partial_k \eta^j].
\end{aligned}$$

This result means that the null spinning  $p$ -brane is a system of rank 1.

In the BFV formalism, the gauge fixing appears associated with the choice of the gauge fermion  $\Psi$ , which in [62] is taken to be

$$\Psi = \int d^p \sigma \left[ \mathcal{P}_J N^J + \bar{b}_J \lambda^J + \frac{1}{\varepsilon} N^j \bar{\eta}_j + \frac{1}{\varepsilon} \lambda^j \bar{C}_j \right], \quad (3.12)$$

where  $\varepsilon$  is an arbitrary parameter that is set to zero at the end of the calculations.

In order to integrate the expression (3.12) one can impose the following boundary conditions

$$\begin{aligned}
x_1^\mu(\underline{\sigma}, \tau_1) &= x^\mu(\underline{\sigma}), & x_2^\mu(\underline{\sigma}, \tau_2) &= x_2^\mu(\underline{\sigma}), \\
\Gamma_J^\mu(\underline{\sigma}, \tau_1) + \Gamma_J^\mu(\underline{\sigma}, \tau_2) &= 2\gamma_J^\mu(\underline{\sigma}), \\
\eta_J(\underline{\sigma}, \tau_1) = \eta_J(\underline{\sigma}, \tau_2) &= 0, & \bar{\eta}_J(\underline{\sigma}, \tau_1) = \bar{\eta}_J(\underline{\sigma}, \tau_2) &= 0, \\
C_J(\underline{\sigma}, \tau_1) = C_J(\underline{\sigma}, \tau_2) &= 0, & \bar{C}_J(\underline{\sigma}, \tau_1) = \bar{C}_J(\underline{\sigma}, \tau_2) &= 0.
\end{aligned} \tag{3.13}$$

Using the Fradkin-Vilkovisky theorem, after some calculations, one obtains

$$\begin{aligned}
Z &= \int \mathcal{D}\mu \exp(iS_{eff}) \\
&= \int \prod_{\underline{\sigma}} dp^\mu(\underline{\sigma}) \frac{\int d^p \sigma \gamma_{\mu 0}(\underline{\sigma}) p^\mu(\underline{\sigma}) \exp(i \int d^p \sigma p^\mu \Delta x_\mu(\underline{\sigma}))}{\int d^p \sigma p^2(\underline{\sigma})},
\end{aligned} \tag{3.14}$$

which is the propagator for a null spinning  $p$ -brane in the momentum space. In receiving (3.14), it is supposed that

$$\Gamma_J^\mu(\underline{\sigma}, \tau) = \gamma_J^\mu(\underline{\sigma}) + \psi_J^\mu(\underline{\sigma}, \tau),$$

where  $\psi_J^\mu$  is a quantum fluctuation that fulfils antiperiodic boundary conditions

$$\psi_J^\mu(\underline{\sigma}, \tau_1) + \psi_J^\mu(\underline{\sigma}, \tau_2) = 0,$$

in accordance with (3.13).

The expression (3.14) shows that the spectrum of the null spinning  $p$ -brane is continuous, so that this model has no critical dimensions. However, a note of caution is necessary: the change of boundary conditions, or equivalently *the change of the operator ordering*, may introduce critical dimensions. More precisely, if we modify the boundary conditions within the ordering prescription for the operators proposed in [83, 84], the critical dimensions that appear for the bosonic null  $p$ -branes for example are

$$D = \frac{5p + 8}{p}.$$

We note that this is in disagreement with the result in [85] for tensile  $p$ -branes, which is

$$D = \frac{5p+8}{p} (p+1),$$

contrary to the author's claim.

Finally, it is pointed out in [62] that it is possible to show that the functional diffusion equation for the null spinning  $p$ -brane reads

$$\frac{\partial G[M, M_0]}{\partial V} = \frac{\delta^2 G[M, M_0]}{\delta x^2(\underline{\sigma})},$$

where  $V$  is the world-volume,  $G[M, M_0]$  is the propagator and  $M, M_0$  represent the final and initial configurations of the null spinning  $p$ -brane respectively.

Now we turn to the considerations devoted to tensionless branes made in the review article [63]. The results of [70, 86, 87, 88, 89] are contained there, so we will not pay separate attention to them. As a matter of fact, [88] and [89] are extended versions of [86, 87].

We start with the *bosonic* null  $p$ -brane case. The corresponding action may be represented in the following simple, reparametrization invariant form [61]

$$S_{0,p} = \frac{1}{2} \int d^{p+1} \xi \frac{\det(\partial_J x^\mu \partial_K x_\mu)}{E}, \quad (3.15)$$

where  $E(\tau, \underline{\sigma})$  is a hyper-sheet density which plays the role of Lagrange multiplier. The action (3.15) is invariant under the following (infinitesimal) conformal transformations

$$\begin{aligned} \delta_D x^\mu &= -\lambda x^\mu, & \delta_D E &= -2(p+1)\lambda E, \\ \delta_c x^\mu &= -c^\mu x^2 - 2(cx)x^\mu, & \delta_c E &= -4(p+1)(cx)E. \end{aligned}$$

For comparison, we write down also the action for the tensile brane in a form in which the limit  $T = (\alpha')^{-(p+1)/2} \rightarrow 0$  can be taken

$$S_p = \frac{1}{2} \int d^{p+1} \xi \left[ \frac{|\det(\partial_J x^\mu \partial_K x_\mu)|}{E} + \frac{1}{(\alpha')^{p+1}} E \right]. \quad (3.16)$$

After substitution of the general solution

$$E = \left( (\alpha')^{p+1} |\det(\partial_J x^\mu \partial_K x_\mu)| \right)^{1/2}$$

of the equation of motion for  $E$  into the action (3.16), the latter takes the Dirac-Nambu form

$$S'_p = \frac{-1}{(\alpha')^{(p+1)/2}} \int d^{p+1}\xi \sqrt{|\det(\partial_J x^\mu \partial_K x_\mu)|}. \quad (3.17)$$

The action (3.15) is characterized by the following constraints

$$T_0 \equiv \eta^{\mu\nu} p_\mu p_\nu \approx 0, \quad T_j \equiv p_\nu \partial_j x^\nu \approx 0,$$

which originate from the arbitrariness in the choice of the world-volume parametrization. Their (equal  $\tau$ ) Poisson bracket algebra reads

$$\begin{aligned} \{T_0(\underline{\sigma}_1), T_0(\underline{\sigma}_2)\} &= 0, \\ \{T_0(\underline{\sigma}_1), T_j(\underline{\sigma}_2)\} &= -[T_0(\underline{\sigma}_1) + T_0(\underline{\sigma}_2)] \partial_j \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\ \{T_j(\underline{\sigma}_1), T_k(\underline{\sigma}_2)\} &= T_j(\underline{\sigma}_2) \partial_k^2 \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2) - (1 \leftrightarrow 2, j \leftrightarrow k). \end{aligned}$$

Let us suppose that there exists a scalar field  $\varphi(x)$ , with vacuum expectation value (VEV)

$$\langle \varphi \rangle \propto M_{Planck} = \left( \frac{hc}{G} \right)^{1/2} \approx 10^{19} GeV/c^2$$

in four dimensional space-time. Then free  $p$ -branes and null  $p$ -branes may be considered as the theories corresponding to different vacuum states of  $p$ -brane interacting with a background scalar field  $\varphi(x)$ . The corresponding action in 4-dimensions may be chosen in the form

$$S_p = \frac{\gamma_p}{2} \int d^{p+1}\xi \left[ \frac{\det(\partial_J x^\mu \partial_K x_\mu)}{E} + \lambda E \varphi^{2p+2} - \frac{1}{\alpha'} E \varphi^{2p} + \dots \right], \quad (3.18)$$

where  $\lambda$  is dimensionless coupling constant. Here the scalar field potential has been chosen in such a way that it coincide with the Higgs potential  $\propto \lambda \varphi^4 - \mu^2 \varphi^2$  when  $p = 1$ .

The VEV's  $\langle \varphi \rangle$  corresponding to different extreme of the potential energy of the field  $\varphi$  are given by the following expressions

$$\langle \varphi \rangle_0 = 0, \quad (3.19)$$

$$\langle \varphi \rangle_\pm = \pm \left( \frac{p}{(p+1)\lambda\alpha'} \right)^{1/2}. \quad (3.20)$$



The equality (3.19) describes the symmetric phase and the equality (3.20) describes the phase with broken conformal and discrete ( $\varphi \rightarrow -\varphi$ ) symmetries. In the neighborhood of the solution (3.19), the action (3.18) coincides with the null  $p$ -brane action (3.15). Correspondingly, in the neighborhood of the solution (3.20), the action (3.18) coincides with the  $p$ -brane action (3.17) after the exclusion of the auxiliary field  $E$ . This qualitative consideration may be viewed as an illustration of a possible mechanism producing nonzero tension for the null  $p$ -brane. This mechanism is similar to the Higgs mechanism and rebuilds the tensionless branes into the Dirac-Nambu  $p$ -branes with tension  $\propto \langle \varphi \rangle^{p+1}$ .

Now let us consider the BRST quantization of the null  $p$ -brane theory and reproduce the result obtained in [63] that there is no critical dimensions.

At first, one studies the equations of motion generated by the action (3.15). The determinant  $\tilde{G} \equiv \det \tilde{G}_{JK}$  of the induced world-volume metric  $\tilde{G}_{JK}$

$$\begin{aligned}\tilde{G}_{JK} &= \partial_J x^\mu \partial_K x_\mu = \begin{pmatrix} \dot{x}^\mu \dot{x}_\mu & \dot{x}^\mu \partial_k x_\mu \\ \partial_j x^\mu \dot{x}_\mu & G_{jk} \end{pmatrix}, \\ G_{jk} &= \partial_j x^\mu \partial_k x_\mu\end{aligned}\tag{3.21}$$

in the integrand of  $S_{0,p}$  may be presented in the form

$$\tilde{G} = G \dot{x}^\mu \Pi_\mu^\nu \dot{x}_\nu, \quad G = \det G_{jk},\tag{3.22}$$

where the matrix  $\Pi_\mu^\nu$  is defined by the relations

$$\Pi_\mu^\nu = \delta_\mu^\nu - \partial_j x_\mu (G^{-1})^{jk} \partial_k x^\nu, \quad \Pi_\mu^\lambda \Pi_\lambda^\nu = \Pi_\mu^\nu.\tag{3.23}$$

The representation (3.22) follows from the well known relation for the determinant of a block matrix

$$\det \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \det (A - BD^{-1}C) \det D.$$

From the variation of the action (3.15) with respect to  $x^\mu$  and  $E$ , one receives the equations of motion

$$2 \left[ \frac{\partial}{\partial \tau} p_\lambda^\tau + \frac{\partial}{\partial \sigma^l} p_\lambda^l \right] = 0,\tag{3.24}$$

$$G \left[ \dot{x}^2 - (\dot{x}_\mu \partial_j x^\mu) (G^{-1})^{jk} (\partial_k x^\nu \dot{x}_\nu) \right] = 0,\tag{3.25}$$

where

$$\begin{aligned}
p_\lambda^\tau &= \frac{G}{E} \left[ \dot{x}_\lambda - \partial_j x_\lambda (G^{-1})^{jk} (\partial_k x^\nu \dot{x}_\nu) \right], \\
p_\lambda^l &= \frac{\dot{x}^2}{2E} \frac{\delta G}{\delta \partial_l x^\lambda} \\
&\quad - \frac{(\dot{x}^\mu \partial_k x_\mu)}{2E} \left[ (G^{-1})^{kj} (\partial_j x^\nu \dot{x}_\nu) \frac{\delta G}{\delta \partial_l x^\lambda} + 2 (G^{-1})^{kl} G \dot{x}_\lambda + G (\dot{x}_\mu \partial_j x^\mu) \frac{\delta (G^{-1})^{jk}}{\delta \partial_l x^\lambda} \right].
\end{aligned}$$

If we fix  $p$  gauge degrees of freedom from  $(p+1)$  ones contained in the reparametrization symmetry group by the following gauge conditions

$$\dot{x}_\mu \partial_j x^\mu = 0, \quad (3.26)$$

then the equations (3.24), (3.25) take a more simple form

$$\frac{\partial}{\partial \tau} \left( \frac{G}{E} \dot{x}_\lambda \right) = 0, \quad G \dot{x}^2 = 0. \quad (3.27)$$

The momentum density of null  $p$ -brane in the gauge (3.26) equals to

$$p_\lambda = \frac{G}{E} \dot{x}_\lambda \quad (3.28)$$

and  $p_\lambda = 0$  if  $G = 0$ , so the solution  $G = 0$  of equations (3.27) corresponds to trivial dynamics. That is why one chooses  $G \neq 0$  and equations (3.27) become equivalent to the equations

$$\frac{\partial}{\partial \tau} \left( \frac{G}{E} \dot{x}_\lambda \right) = 0, \quad \dot{x}^2 = 0. \quad (3.29)$$

Taking into account that additional gauge condition may be added to (3.26) the authors choose this condition in the form

$$\frac{\partial}{\partial \tau} \left( \frac{G}{E} \right) = 0. \quad (3.30)$$

Then the equations of motion (3.29) take a linear form

$$\ddot{x}^\mu = 0, \quad \dot{x}^2 = 0. \quad (3.31)$$

Of course, the variation of the action (3.15) have to be supplemented with appropriate boundary conditions for open or closed null  $p$ -branes. In what follows, the discussion is restricted to the closed case only.

The following step is the quantization of null bosonic  $p$ -brane. In order to simplify the technical aspects of the BRST-quantization procedure, the gauge condition

$$\frac{G}{E} = \gamma_p = \text{const} \quad (3.32)$$

is used instead of (3.30). This gauge gives the possibility to consider the minimal extension of the original phase space. It may be constructed by extending the initial phase space with the canonically conjugated Grassmannian ghost pairs  $(\eta^J, \mathcal{P}_J)$  for the constraints  $T_J$ .

So, the classical BRST charge [79, 80, 90, 82] for the closed null  $p$ -brane in the minimal sector has the following form [70]

$$\Omega^{min} = \int d^p \sigma \{ T_0 \eta^0 + T_j \eta^j + \mathcal{P}_0 [(\partial_j \eta^0) \eta^j - \eta^0 (\partial_j \eta^j)] - \mathcal{P}_k \eta^j (\partial_j \eta^k) \}. \quad (3.33)$$

It is easy to see that the BRST charge (3.33) satisfy the equation

$$\{\Omega^{min}, \Omega^{min}\} = 0. \quad (3.34)$$

In the quantum case the charge  $\Omega^{min}$  transforms into the quantum BRST operator  $\hat{\Omega}^{min}$ . Then the nilpotency condition

$$\left(\hat{\Omega}^{min}\right)^2 = 0 \quad (3.35)$$

ought to be fulfilled for the self consistency of the quantum theory. However, anomalies can appear in the right hand side of equation (3.35) as a result of quantum ordering process. Taking into account the point-like character of the first equation in (3.31) for the null  $p$ -brane it is natural to choose the initial data for ordinary phase space variables  $\hat{q}_0$  and  $\hat{p}_0$  as the "physical variables" in terms of which quantum ordering must be done. Remind that in the string case quantum ordering in terms of the creation and annihilation oscillator operators  $a$  and  $a^\dagger$  is more preferable as a consequence of the oscillator character of the string equation of motion

$$\ddot{x}^\mu(\tau, \sigma) - x''^\mu(\tau, \sigma) = 0.$$

Thus one chooses the  $\hat{q}_0 \hat{p}_0$ -ordering in the BRST generator  $\hat{\Omega}^{min}$ . This ordering does not give rise to anomalous terms in the calculation of  $\left(\hat{\Omega}^{min}\right)^2$ . As a consequence, the quantum nilpotency condition is satisfied.

Let us discuss the case of  $\hat{q}\hat{p}$ -ordering, which is formulated in terms of the hole operators of phase variables. Actually,  $\hat{\Omega}^{min}$  contains only such combinations of the canonically conjugated operators  $\hat{q}$  and  $\hat{p}$ , for which the ordering is preserved in the process of the anticommutator  $\{\hat{\Omega}^{min}, \hat{\Omega}^{min}\}$  calculation, and it is not difficult to prove this [63].

After transition from  $\hat{q}$ ,  $\hat{p}$  to their initial data  $\hat{q}_0$ ,  $\hat{p}_0$  the considered quantum  $\hat{q}_0, \hat{p}_0$ -ordering also will be conserved. To clear up this statement it is enough to analyze the equations of motion for the ghost sector only, because the equations for  $\hat{x}^\mu$  and  $\hat{p}_\mu$  are linear ones. The BRST-invariant Hamiltonian which generates these equations of motion depends on the choice of the gauge fermion  $\Psi$ , which in the minimal sector has the general form

$$\Psi = \int d^p \sigma \left( \frac{\lambda^0}{2} \mathcal{P}_0 - \zeta^k \mathcal{P}_k \right).$$

Using the arbitrariness in the choice of the Lagrange multipliers  $\lambda^0$  and  $\zeta^k$ , they are chosen in [63] as

$$\lambda^0 = \gamma_p^{-1}, \quad \zeta^k = 0, \quad \Rightarrow \Psi_0 = \frac{1}{2\gamma_p} \int d^p \sigma \mathcal{P}_0.$$

The corresponding Hamiltonian is obtained to be

$$H_{\Psi_0} = -\frac{1}{2\gamma_p} \int d^p \sigma \left( p^\mu p_\mu + \mathcal{P}_0 \partial_k \eta^k \right).$$

The equations of motion  $\dot{f} = \{f, H_{\Psi_0}\}$  generated by  $H_{\Psi_0}$  have the form

$$\begin{aligned} \dot{x}^\mu &= \frac{1}{\gamma_p} p^\mu, & \dot{p}^\mu &= 0, & \dot{\eta}^0 &= \frac{1}{2\gamma_p} \partial_k \eta^k, \\ \dot{\eta}^k &= 0, & \dot{\mathcal{P}}_0 &= 0, & \dot{\mathcal{P}}_k &= \frac{1}{2\gamma_p} \partial_k \mathcal{P}_0. \end{aligned}$$

The general solution of this system of equations is

$$\begin{aligned} x^\mu(\tau, \underline{\sigma}) &= x_0^\mu(\underline{\sigma}) + \frac{\tau}{\gamma_p} p_0^\mu(\underline{\sigma}), & p_\mu(\tau, \underline{\sigma}) &= p_{0\mu}(\underline{\sigma}), \\ \eta^0(\tau, \underline{\sigma}) &= \eta_0^0(\underline{\sigma}) + \frac{\tau}{2\gamma_p} \partial_k \eta_0^k(\underline{\sigma}), & \mathcal{P}_0(\tau, \underline{\sigma}) &= \mathcal{P}_{00}(\underline{\sigma}), \\ \eta^k(\tau, \underline{\sigma}) &= \eta_0^k(\underline{\sigma}), & \mathcal{P}_k(\tau, \underline{\sigma}) &= \mathcal{P}_{0k}(\underline{\sigma}) + \frac{\tau}{2\gamma_p} \partial_k \mathcal{P}_0. \end{aligned}$$

We see that the transition from the  $\hat{q}, \hat{p}$  variables to their initial data  $\hat{q}_0, \hat{p}_0$  is a linear transformation. Owing to this fact the proof of nilpotency condition for  $\hat{\Omega}^{min}$  carried out in the terms of  $\hat{q}\hat{p}$ -ordering does not change after transition to  $\hat{q}_0\hat{p}_0$ -ordering.

Here in [63] the conclusion is drawn that the critical dimensions are absent in the null bosonic  $p$ -brane theory and this theory is quantum mechanically self consistent in flat space-time of arbitrary dimension. However, it is noted that the question about presence or absence of a critical dimension in the quantum theory essentially depends on the choice of initial operator set in terms of which the procedure of operator ordering is defined.

Finally, let us describe the dynamics of null  $p$ -brane in the light cone gauge. At first we note that in the gauge (3.26) the induced metric  $\tilde{G}_{JK}$  takes the form

$$\tilde{G}_{JK}(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & G_{JK}(\xi) \end{pmatrix}. \quad (3.36)$$

The representation is invariant under diffeomorphisms

$$\sigma^j(\tau', \underline{\sigma}') = \sigma^j(\underline{\sigma}'). \quad (3.37)$$

The gauge condition (3.32), which linearizes the null  $p$ -brane equations of motion, is conserved under the diffeomorphism transformations

$$\tau(\tau', \underline{\sigma}') = a(\underline{\sigma}') + \tau' \det \left( \frac{\partial \sigma^j(\underline{\sigma}')}{\partial \sigma'^k} \right). \quad (3.38)$$

The equations of motion  $\ddot{x}^\mu = 0$  are invariant under the transformations (3.37) and (3.38). Therefore, it is possible to use the gauge symmetry (3.37), (3.38) for fixing light-cone gauge condition defined as

$$x^+(\tau, \underline{\sigma}) = c\tau, \quad c = \frac{p_0^+}{\gamma} = \frac{P_0^+}{N_p \gamma}, \quad P_0^+ = \int d^p \sigma p_0^+(\underline{\sigma}). \quad (3.39)$$

This gauge condition is conserved under the diffeomorphisms (3.37) and (3.38) restricted by the condition  $\tau' = \tau$ . We see that the gauge conditions (3.26), (3.32) and (3.39) are characterized by a residual gauge symmetry comprising the so called area-preserving transformations, which are defined by the infinitesimal relations

$$\delta\tau = 0, \quad \delta\sigma^{j_1} = \varepsilon^{j_1 j_2 \dots j_p} \partial_{j_2} \Lambda_{j_3 \dots j_p}.$$

In the light-cone gauge, the general solution of (3.31) is

$$\begin{aligned}x^m(\tau, \underline{\sigma}) &= x_0^m(\underline{\sigma}) + p_0^m(\underline{\sigma}) \gamma_p^{-1} \tau, & (m = 1, \dots, D-1) \\x^-(\tau, \underline{\sigma}) &= x_0^-(\underline{\sigma}) + \frac{N_p}{2\gamma P_0^+} p_{0m}^2(\underline{\sigma}) \\p_0^-(\underline{\sigma}) &= \frac{N_p}{2P_0^+} p_{0m}^2(\underline{\sigma}).\end{aligned}$$

Then for the mass operator of null  $p$ -brane defined by the relation

$$M^2 \equiv P_0^\mu P_{0\mu} \equiv 2P_0^+ P_0^- - P_{0m}^2, \quad P_{0\mu} = \int d^p \sigma p_{0\mu}(\underline{\sigma})$$

we get the representation

$$M^2 = N_p \int d^p \sigma p_{0m}^2(\underline{\sigma}) - \left( \int d^p \sigma p_{0m}(\underline{\sigma}) \right)^2.$$

It can be shown that  $M^2 \geq 0$  and it is independent on the center of mass variables  $q_{0m}, P_{0m}$ .

The canonical Hamiltonian in the light-cone gauge has the form

$$H_{lc} = \frac{P_0^+}{\gamma N_p} \int d^p \sigma p_0^-(\underline{\sigma}) = \frac{1}{2\gamma} \int d^p \sigma p_{0m}^2(\underline{\sigma}) = \frac{1}{2\gamma N_p} [p_{0m}^2(\underline{\sigma}) + M^2]. \quad (3.40)$$

The eigenfunctions  $\Psi_{|k_{0m}; \{k_{ma}\} >}$  associated with the Hamiltonian (3.40) are generalized "plane waves"

$$\Psi_{|k_{0m}; \{k_{ma}\} >} = \exp \left( i \sum_a k_{ma} x_m^a \right) = \exp(i k_{0m} q_{0m}) \exp \left( i \sum_{a \neq 0} k_{ma} x_m^a \right), \quad (3.41)$$

which describe the coherent motion of infinite number of quasi particles. Here the index  $a$  marks a complete orthonormal basis of functions  $Y_a(\underline{\sigma})$  on the (null)  $p$ -brane [63, 91, 92]. The physical subspace can be extracted from the set of vectors (3.41) by the annihilation conditions

$$\sum_{a,b \neq 0} f_{abc}^{de} x_m^a P_m^b \Psi^{phys} = 0.$$

Then, supposing that there exist nonzero solutions of the above equalities, it is not difficult to show that *the spectrum of null  $p$ -brane is continuous* [63].

Now we turn to the null super  $p$ -brane case in four dimensions [63]. The dynamics of null super  $p$ -brane is described by the action (3.11). However, the covariant

quantization in this formulation is a very complicated problem because of an infinite reducibility of the fermionic constraints analogous to those of the superstring. In order to covariantly quantize the null super  $p$ -brane theory in  $D = 4$ , a new twistor-like Lorentz harmonic formulation has been proved to be effective [86, 87]. It uses the Lorentz harmonic superspace

$$z^M = (x^\mu, \theta_A^\alpha, \bar{\theta}^{\dot{\alpha}A}, v_\alpha^\mp, \bar{v}_\alpha^\pm), \quad \alpha = 1, 2; \quad A = 1, \dots, N$$

as a target space. This space is an extension of the usual ( $N$ -extended) superspace  $(x^\mu, \theta_A^\alpha, \bar{\theta}^{\dot{\alpha}A})$  obtained by adding the commuting spinor variables  $(v_\alpha^\mp, \bar{v}_\alpha^\pm)$ , which coincide with the Newman-Penrose diades [93] in four dimensions. In this formulation, the action for null super  $p$ -brane theory is

$$S_{0,p} = -\frac{1}{2} \int d^p \xi \varrho^{(-|+)^J} v_\alpha^- \bar{v}_\alpha^+ \omega_J^{\dot{\alpha}\alpha}, \quad (3.42)$$

where  $\varrho^{(-|+)^J}$  is a world-volume density,

$$\omega_J^{\dot{\alpha}\alpha} = \tilde{\sigma}_\mu^{\dot{\alpha}\alpha} \left( \partial_J x^\mu - i \partial_J \theta_A^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}A} + i \theta_A^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_J \bar{\theta}^{\dot{\alpha}A} \right) \equiv \omega_J^\mu \tilde{\sigma}_\mu^{\dot{\alpha}\alpha}$$

is the invariant Cartan form for ordinary superspace and  $v_\alpha^\mp, \bar{v}_\alpha^\pm$  are restricted by the conditions

$$\Xi(\tau, \underline{\sigma}) \equiv v^{\alpha-}(\tau, \underline{\sigma}) v_\alpha^+(\tau, \underline{\sigma}) - 1 = 0,$$

$$\bar{\Xi}(\tau, \underline{\sigma}) \equiv \bar{v}^{\dot{\alpha}+}(\tau, \underline{\sigma}) \bar{v}_{\dot{\alpha}}^-(\tau, \underline{\sigma}) - 1 = 0.$$

The action (3.42) is invariant under the following local  $\kappa$ -transformations

$$\delta \theta_{\alpha A} = i p_{\alpha\dot{\alpha}} \bar{\kappa}_A^{\dot{\alpha}}, \quad \delta \bar{\theta}_{\dot{\alpha}}^A = -i \kappa^{\alpha A} p_{\alpha\dot{\alpha}},$$

$$\delta x_{\alpha\dot{\alpha}} = 2 \left( \kappa^{\beta A} p_{\beta\dot{\alpha}} \theta_{\alpha A} + \bar{\theta}_{\dot{\alpha}}^A p_{\alpha\dot{\gamma}} \bar{\kappa}_A^{\dot{\gamma}} \right),$$

$$\delta \varrho^J = 0, \quad \delta v_\alpha^\mp = 0, \quad \delta \bar{v}_\alpha^\pm = 0,$$

where

$$p_{\alpha\dot{\alpha}} = \varrho^0(\tau, \underline{\sigma}) v_\alpha^- \bar{v}_{\dot{\alpha}}^+$$

is the momentum density of null super  $p$ -brane.

The variation of the action (3.42) with respect to  $x^\mu$ ,  $\theta^{\alpha A}$ ,  $\bar{\theta}^{\dot{\alpha} A}$ ,  $v_\alpha^-$  and  $\bar{v}_\alpha^+$  leads to the equations

$$\begin{aligned}\partial_J \left( \varrho^J v_\alpha^- \bar{v}_\alpha^+ \right) &= 0 \quad \text{or} \quad \dot{p}_{\alpha\dot{\alpha}} = \partial_j \left( \varrho^j v_\alpha^- \bar{v}_\alpha^+ \right), \\ \varrho^J \partial_J \theta_{\alpha A} &= \zeta_A(\tau, \underline{\varrho}) v_\alpha^-, \quad \varrho^J \partial_J \bar{\theta}_{\dot{\alpha}}^A = \bar{\zeta}^A(\tau, \underline{\varrho}) \bar{v}_\alpha^+, \\ \varrho^J \omega_{J\alpha\dot{\alpha}} &= e(\tau, \underline{\varrho}) v_\alpha^- \bar{v}_\alpha^+ \quad \text{or} \quad \omega_{0\alpha\dot{\alpha}} = \frac{e}{(\varrho^0)^2} p_{\alpha\dot{\alpha}} - \frac{\varrho^j}{\varrho^0} \omega_{j\alpha\dot{\alpha}}.\end{aligned}$$

Finally, the  $\varrho^J$ -variation generates the constraints

$$v_\alpha^- \bar{v}_\alpha^+ \omega_J^{\dot{\alpha}\alpha} \approx 0 \quad \text{or} \quad p_{\alpha\dot{\alpha}} p^{\dot{\alpha}\alpha} \approx 0, \quad p_{\alpha\dot{\alpha}} \omega_j^{\dot{\alpha}\alpha} \approx 0.$$

The arbitrary functions  $\varrho^J(\tau, \underline{\varrho})$ ,  $\zeta_A(\tau, \underline{\varrho})$ ,  $\bar{\zeta}^A(\tau, \underline{\varrho})$  and  $e(\tau, \underline{\varrho})$  in the above equations show the invariance of the action (3.42) under reparametrizations and  $\kappa$ -transformations.

It can be proved [86, 87, 63] that the presence of the auxiliary variables  $v_\alpha^\mp$ ,  $\bar{v}_\alpha^\pm$  and  $\varrho^{(-|+)^J}$  in (3.42) does not increase the number of physical degrees of freedom and the formulations (3.11), (3.42) for null super  $p$ -brane theory are equivalent at the classical level.

However, the harmonics  $v_\alpha^\mp$  and their complex conjugated  $\bar{v}_\alpha^\pm$  play an exceptional role in the Lorentz covariant division of null super  $p$ -brane constraints into irreducible once of first and second class. This is carried out by projecting the Grassmann constraints

$$D_\alpha^A = -\pi_\alpha^A + i p_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha} A}$$

into first class  $D^{-A} = v^{\alpha-} D_\alpha^A$  and second class  $D^{+A} = v^{\alpha+} D_\alpha^A$  constraints.

In terms of the coordinates  $(z^M, \varrho^{(-|+)^J})$  and their conjugated momenta  $(p_M, p_J^e) = (p_\mu, \pi_\alpha^A, \bar{\pi}_{\dot{\alpha} A}; p_\alpha^\pm, \bar{p}_{\dot{\alpha}}^\mp, p_J^e)$ , the first class constraints

$$Y_\Lambda = (Y_{\Lambda'}, T_j) = \left( D^{-A}, \bar{D}_A^+, p^{+-}, p_j^e, \nabla^0, \bar{\nabla}^0, \nabla^{-2}, \bar{\nabla}^{+2}, T_j \right)$$

of null super  $p$ -brane theory have the form

$$D^{-A} \equiv v^{\alpha-} D_\alpha^A \equiv v^{\alpha-} \left( -\pi_\alpha^A + i p_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha} A} \right) \approx 0,$$



$$\begin{aligned}
p^{+-} &\equiv v^{\alpha-} \sigma_{\alpha\dot{\alpha}}^{\mu} p_{\mu} \approx 0, & p_j^e &\approx 0, \\
\nabla^0 &\equiv v^{\alpha+} p_{\alpha}^{-} - v^{\alpha-} p_{\alpha}^{+} + \varrho^{0(-|+)} p_j^e \approx 0, & \nabla^{-2} &\equiv v^{\alpha-} p_{\alpha}^{-} \approx 0, \\
T_j &\equiv \frac{1}{2} \omega_j^{\alpha\dot{\alpha}} p_{\alpha\dot{\alpha}} + \left( \partial_j \theta_A^{\alpha} D_{\alpha}^A + \partial_j v^{\alpha\mp} p_{\alpha}^{\pm} + c.c. \right) - \partial_j \varrho^{0(-|+)} p_j^e \approx 0,
\end{aligned}$$

and complex conjugated to  $D^{-A}$ ,  $\nabla^0$ ,  $\nabla^{-2}$  constraints  $\bar{D}_A^{+}$ ,  $-\bar{\nabla}^0$ ,  $\bar{\nabla}^{+2}$  (**Remark:** there are obvious mistakes in the expressions for  $\nabla^0$  and  $T_j$ ). They form the following Poisson bracket algebra (we list the nonzero brackets only)

$$\begin{aligned}
\{T_j(\underline{\sigma}_1), T_k(\underline{\sigma}_2)\} &= T_j(\underline{\sigma}_2) \partial_k^2 \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2) - (1 \leftrightarrow 2, j \leftrightarrow k), \\
\{T_j(\underline{\sigma}_1), Y_{\Lambda'}(\underline{\sigma}_2)\} &= -Y_{\Lambda'}(\underline{\sigma}_1) \partial_j^2 \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\
\{\nabla^0(\underline{\sigma}_1), Y_{\Lambda'}(\underline{\sigma}_2)\} &= q_R(Y_{\Lambda'}) \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\
\{\bar{\nabla}^0(\underline{\sigma}_1), Y_{\Lambda'}(\underline{\sigma}_2)\} &= q_L(Y_{\Lambda'}) \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\
\{D^{-A}(\underline{\sigma}_1), \bar{D}_B^{+}(\underline{\sigma}_2)\} &= -2ip^{+-} \delta_B^A \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2),
\end{aligned} \tag{3.43}$$

where  $q_{L,R}(Y_{\Lambda'})$  are the charges of  $Y_{\Lambda'}$  under  $U_{L,R}(1)$  symmetry groups.

The constraints  $p^{+-}$  and  $T_j$  correspond to reparametrization symmetry of the action (3.42),  $D^{-A}$  and  $\bar{D}_A^{+}$  describe the fermionic  $\kappa$ -symmetry, whereas  $\nabla^0$  and  $\bar{\nabla}^0$  are related to the local  $U_L(1) \times U_R(1) \cong SO(1,1) \times SO(2)$  symmetry and  $\nabla^{-2}$ ,  $\bar{\nabla}^{+2}$  are connected with the local shifts of  $v^{\alpha+}$  and  $\bar{v}^{\dot{\alpha}-}$ .

The second class constraints  $S_f$  in the twistor-like formulation (3.42) appear to be the pairs

$$\begin{aligned}
\Xi &\approx 0, \\
\chi &\equiv v^{\alpha-} p_{\alpha}^{+} + v^{\alpha+} p_{\alpha}^{-} + \varrho^0 p_0^e \approx 0, \\
\bar{\Xi} &\approx 0, \\
\bar{\chi} &\approx 0, \\
D^{+A} &\equiv v^{\alpha+} D_{\alpha}^A \equiv v^{\alpha+} \left( -\pi_{\alpha}^A + ip_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}A} \right) \approx 0, \\
\bar{D}^{-A} &\equiv \bar{v}^{\dot{\alpha}-} \bar{D}_{\dot{\alpha}A} \equiv \bar{v}^{\dot{\alpha}-} \left( \bar{\pi}_{\dot{\alpha}A} - i\theta_A^{\alpha} p_{\alpha\dot{\alpha}} \right) \approx 0,
\end{aligned}$$

$$\begin{aligned}
\nabla^{+2} &\equiv v^{\alpha+} p_{\alpha}^{+} \approx 0, \\
p^{--} &\equiv -v^{\alpha-} \sigma_{\alpha\dot{\alpha}}^{\mu} \bar{v}^{\dot{\alpha}-} p_{\mu} \approx 0,
\end{aligned}$$

$$\begin{aligned}\bar{\nabla}^{-2} &\equiv \bar{v}^{\dot{\alpha}-} \bar{p}_{\dot{\alpha}}^- \approx 0, \\ p^{++} &\equiv -v^{\alpha+} \sigma_{\alpha\dot{\alpha}}^{\mu} \bar{v}^{\dot{\alpha}+} p_{\mu} \approx 0,\end{aligned}$$

$$\begin{aligned}p_0^g &\approx 0 \quad , \\ \varrho^{0(-|+)} - p^{(-|+)} &\equiv \varrho^{0(-|+)} - v^{\alpha+} \sigma_{\alpha\dot{\alpha}}^{\mu} \bar{v}^{\dot{\alpha}-} p_{\mu} \approx 0.\end{aligned}$$

The Poisson bracket matrix of the constraints  $S_f$  may be transformed to block-diagonal form if certain redefinitions are performed. The following step is to use the method for conversion of second class constraints into effective first class abelian  $\mathcal{A}_f$  ones [94, 95, 96]. The conversion procedure suggests the introduction of new additional set of canonically conjugated variables  $(q^R, p_R)$ , so that any definite pair of the original second class constraints corresponds to some pair of the newly introduced phase space variables. Now the transition from the initial first class constraints  $Y_{\Lambda}$  to the effective ones  $\tilde{Y}_{\Lambda}$ , depending also on  $(q^R, p_R)$ , must be done. However, it turns out that the resulting algebra essentially complicates the evaluation procedure for the BRST generator  $\Omega$ . For this reason, a new set of effective first class constraints  $\tilde{\tilde{Y}}_{\Lambda}$  have been introduced in [63]. Their algebra coincides with the original  $Y_{\Lambda}$  algebra (3.43) with the exception of the Poisson brackets

$$\{\tilde{\tilde{\nabla}}^{-2}(\underline{\sigma}_1), \tilde{\tilde{\nabla}}^{+2}(\underline{\sigma}_2)\} = \mathcal{E}(\underline{\sigma}_1) \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2),$$

where

$$\mathcal{E} = \frac{i}{2J} \left[ \tilde{\tilde{D}}^{-A} \tilde{\tilde{D}}_A^+ + \frac{2}{i} \tilde{\tilde{p}}^{+-} \left( \tilde{\tilde{\nabla}}^0 + \tilde{\tilde{\nabla}}^0 \right) \right]$$

is a first class constraint (**Remark:** the quantity  $J$  is not defined in [63]). As a result, the total algebra of the *effective first class constraints* for null super  $p$ -brane is found to be

$$\{\mathcal{A}_f, \mathcal{A}_g\} = 0, \quad \{\tilde{\tilde{Y}}_{\Lambda}, \mathcal{A}_f\} = 0, \quad \{\tilde{\tilde{Y}}_{\Lambda}, \tilde{\tilde{Y}}_{\Sigma}\} = C_{\Lambda\Sigma}^{\Pi} \tilde{\tilde{Y}}_{\Pi} + \mathcal{E}_{\Lambda\Sigma}, \quad (3.44)$$

with the same structure constants  $C_{\Lambda\Sigma}^{\Pi}$  as in the original algebra (3.43) and

$$\mathcal{E}_{\Lambda\Sigma} = \left[ \delta_{\tilde{\tilde{Y}}_{\Lambda}, \tilde{\tilde{\nabla}}^{-2}} \delta_{\tilde{\tilde{Y}}_{\Sigma}, \tilde{\tilde{\nabla}}^{+2}} - (\Lambda \leftrightarrow \Sigma) \right] \mathcal{E}.$$

The classical BRST charge for the algebra (3.44) in the minimal sector equals to  $\Omega_{min} = \Omega' + \mathcal{A}_f c''^f$  with  $\Omega'$  defined as

$$\begin{aligned} \Omega' = & \int d^p \sigma [Y_\Lambda^{Mod} c^\Lambda - 2i\pi^{+-} c_A^+ \bar{c}^{-A} - \frac{i}{J} (\pi^0 + \bar{\pi}^0) \tilde{p}^{+-} \zeta^{+2} \bar{\zeta}^{-2} \\ & + \left( \frac{i}{4J} p^{-A} \tilde{\bar{D}}_A^+ \zeta^{+2} \bar{\zeta}^{-2} + c.c. \right) + \left( \frac{1}{2J} \pi^{+-} \bar{p}_A^+ \bar{c}^{-A} \zeta^{+2} \bar{\zeta}^{-2} + c.c. \right)]. \end{aligned} \quad (3.45)$$

The modified constraints  $Y_\Lambda^{Mod}$  entering into (3.45) coincide with  $\tilde{Y}_\Lambda$  except the constraints  $T_j^{Mod}, \nabla^{0Mod}, \bar{\nabla}^{0Mod}$ .

In order to construct the quantum BRST operator

$$\hat{\Omega}_{min} = \hat{\Omega}' + \hat{\mathcal{A}}_f \hat{c}''^f,$$

a generalized  $\hat{q}\hat{p}$ -ordering is chosen in [63]. The term "generalized" means that one may include some momentum variables into a  $\hat{q}$ -set and some coordinate variables into a  $\hat{p}$ -set. Then, in analogy with the bosonic null  $p$ -brane case, it is proven that this ordering is preserved in the process of computation of the anticommutator  $\{\hat{\Omega}', \hat{\Omega}'\}$  and this leads to the fulfillment of the condition  $(\hat{\Omega}')^2 = 0$ . This consideration permits to the authors of [63] to conclude that the null super  $p$ -brane theories are self-consistent in  $D = 4$ .

A number of classically equivalent actions for  $p$ -branes are derived and their tensionless limits are discussed in [64, 65]. The starting point is the Nambu-Goto-Dirac world-volume action

$$S = T \int d^{p+1} \xi \sqrt{-\det \gamma_{JK}} \quad (3.46)$$

where  $X^\mu = X^\mu(\xi)$  and

$$\gamma_{JK} \equiv \partial_J X^\mu \partial_K X^\nu \eta_{\mu\nu} \quad (3.47)$$

is the metric induced on the world-volume from the Minkowski space-time metric  $\eta_{\mu\nu}$ . The generalized momenta derived from the Lagrangian in (3.46) are

$$P_\mu = T \sqrt{-\gamma} \gamma^{J0} \partial_J X_\mu. \quad (3.48)$$

where  $\gamma^{JK}$  is the inverse of  $\gamma_{JK}$ . They satisfy the constraints

$$P^2 + T^2 \gamma \gamma^{00} = 0$$

$$P_\mu \partial_a X^\mu = 0, \quad a = 1, \dots, p. \quad (3.49)$$

Here  $\gamma \equiv \det \gamma_{JK}$ . As usual for a diffeomorphism invariant theory, the naive Hamiltonian vanishes and the total Hamiltonian consists of the sum of the constraints (3.49) multiplied by Lagrange multipliers, which we shall call  $\lambda$  and  $\rho^a$ :

$$\mathcal{H} = \lambda(P^2 + T^2 \gamma \gamma^{00}) + \rho^a P \cdot \partial_a X \quad (3.50)$$

The phase space action thus becomes

$$S^{PS} = \int d^{p+1} \xi \left\{ P \cdot \dot{X} - \lambda(P^2 + T^2 \gamma \gamma^{00}) - \rho^a P \cdot \partial_a X \right\}. \quad (3.51)$$

One integrates out the momenta to find the configuration space action

$$S^{CS} = \frac{1}{2} \int d^{p+1} \xi \frac{1}{2\lambda} \left\{ \dot{X}^2 - 2\rho^a \dot{X}^\mu \partial_a X_\mu + \rho^a \rho^b \partial_b X^\mu \partial_a X_\mu - 4\lambda^2 T^2 \gamma \gamma^{00} \right\}. \quad (3.52)$$

For  $p = 1$ , the following identification may be done

$$g^{JK} = \begin{pmatrix} -1 & \rho \\ \rho & -\rho^2 + 4\lambda^2 T^2 \end{pmatrix}, \quad (3.53)$$

which leads to the usual Weyl invariant tensile string action

$$S = -\frac{1}{2} T \int d^2 \xi \sqrt{-g} g^{JK} \partial_J X^\mu \partial_K X^\nu \eta_{\mu\nu}. \quad (3.54)$$

For  $p > 1$  it is not possible to directly identify the geometric fields in (3.52). One first has to rewrite it as

$$S^{CS} = \frac{1}{2} \int d^{p+1} \xi \left\{ \frac{h^{JK} \gamma_{JK}}{2\lambda} - 2\lambda T^2 G(p-1) + 2\lambda T^2 G G^{ab} \gamma_{ab} \right\}, \quad (3.55)$$

where

$$h^{JK} = \begin{pmatrix} 1 & -\rho^a \\ -\rho^a & \rho^a \rho^b \end{pmatrix} \quad (3.56)$$

is a rank 1 auxiliary matrix and  $G_{ij}$  is a  $p$ -dimensional auxiliary metric with determinant  $G$ . (Integrating out  $G_{ij}$  one recovers (3.52).) Now the identification

$$g^{JK} = \frac{1}{4} T^{-2} \lambda^{-2} G^{-1} \begin{pmatrix} -1 & \rho^a \\ \rho^a & -\rho^a \rho^b + 4\lambda^2 T^2 G G^{ab} \end{pmatrix} \quad (3.57)$$

produces the usual  $p$ -brane action involving the world-volume metric  $g_{JK}$ :

$$S = -\frac{1}{2}T \int d^{p+1}\xi \sqrt{-g} \left\{ g^{JK} \partial_J X^\mu \partial_K X^\nu \eta_{\mu\nu} - (p-1) \right\}. \quad (3.58)$$

The identification (3.57) tells us the transformation properties of the Lagrange multipliers. Note that for  $p = 0, 1$  the auxiliary metric  $G_{ij}$  never appears in (3.55), and the configuration space action is the usual manifestly reparametrization invariant massive point-particle action and reparametrization invariant tensile string action respectively.

It is clear from the above procedure that we may take the limit  $T \rightarrow 0$  anywhere between (3.50) and (3.57). The identification (3.57) will differ in that limit, however. The metric density  $T\sqrt{-g}g^{JK}$  becomes degenerate and gets replaced by a rank 1 matrix which can be written as  $V^J V^K$  in terms of the vector density  $V^J$

$$V^J \leftrightarrow \frac{1}{\sqrt{2}\lambda} (1, \rho^a). \quad (3.59)$$

In fact, using this prescription the  $T \rightarrow 0$  limit of the  $p$ -brane action is

$$S = \int d^{p+1}\xi V^J V^K \partial_J X^\mu \partial_K X^\nu \eta_{\mu\nu}. \quad (3.60)$$

Another type of null bosonic  $p$ -brane action is given in [65]

$$S = \int d^{p+1}\xi \left( \Delta \cdot \mathbf{P} - \frac{1}{2} V(\xi) \mathbf{P} \cdot \mathbf{P} \right),$$

where  $\Delta^{\mu_0 \dots \mu_p}$  is the following antisymmetric space-time tensor

$$\Delta^{\mu_0 \dots \mu_p} = \frac{1}{[(p+1)!]^{1/2}} \epsilon^{J_0 \dots J_p} \partial_{J_0} x^{\mu_0} \dots \partial_{J_p} x^{\mu_p}$$

and  $\mathbf{P}_{\mu_0 \dots \mu_p}$  are totally antisymmetric "generalized momenta" satisfying the constraint  $\mathbf{P}^2 = -T^2$ .

The quantization of different types of tensionless  $p$ -branes is also discussed in [66]. Now we are going to describe the results of this paper. The constraints are written there in the form

$$\begin{aligned} \phi^{-1}(\sigma_1, \dots, \sigma_p) &= P^\mu P_\mu(\sigma_1, \dots, \sigma_p) = 0 \\ L^\alpha(\sigma_1, \dots, \sigma_p) &= P^\mu \partial_\alpha X_\mu(\sigma_1, \dots, \sigma_p) = 0, \end{aligned}$$

Note that the greek index  $\alpha$  runs from 1 to  $p$ . In Fourier modes the constraints read (for simplicity closed  $p$ -branes are considered)

$$\phi_{m_1, \dots, m_p}^{-1} = \frac{1}{2} \sum_{k_1, \dots, k_p = -\infty}^{+\infty} p_{m_1 - k_1, \dots, m_p - k_p} \cdot p_{k_1, \dots, k_p} = 0, \quad (3.61)$$

$$L_{m_1, \dots, m_p}^\alpha = -i \sum_{k_1, \dots, k_p = -\infty}^{+\infty} k_\alpha p_{m_1 - k_1, \dots, m_p - k_p} \cdot x_{k_1, \dots, k_p} = 0 \quad (3.62)$$

and they satisfy the following algebra

$$[\phi_{m_1, \dots, m_p}^{-1}, L_{n_1, \dots, n_p}^\alpha] = (m_\alpha - n_\alpha) \phi_{m_1 + n_1, \dots, m_p + n_p}^{-1}, \quad (3.63)$$

$$\begin{aligned} [L_{m_1, \dots, m_p}^\alpha, L_{n_1, \dots, n_p}^\beta] &= m_\beta L_{m_1 + n_1, \dots, m_p + n_p}^\alpha - n_\alpha L_{m_1 + n_1, \dots, m_p + n_p}^\beta \\ &\quad + A^{\alpha\beta}(m_1, \dots, m_p) \delta_{m_1 + n_1} \dots \delta_{m_p + n_p}. \end{aligned} \quad (3.64)$$

The right hand side of equation (3.64), when  $m_1 + n_1 = \dots = m_p + n_p = 0$ , is expressed in terms of  $L_{0, \dots, 0}^\alpha$ . But this operator is not well defined since it depends on the different orderings of  $x_{m_1, \dots, m_p}^\mu$  and  $p_{m_1, \dots, m_p}^\mu$ . Taking into account this ambiguity the possible central extensions in the right hand side of the commutators (3.64) are included. The values of these central extensions are constrained by the Jacobi identities and the antisymmetry of the commutators. It is found that for  $p > 1$

$$A^{\alpha\beta}(m_1, \dots, m_p) = A^{\alpha\beta}(m_\alpha, m_\beta) = \frac{1}{2} (m_\beta d^\alpha + m_\alpha d^\beta),$$

where  $d^\alpha$  are constants.

In order to clarify the implications of the last relation one takes  $\alpha = \beta$  in (3.64).

Then

$$[L_{m_1, \dots, m_p}^\alpha, L_{n_1, \dots, n_p}^\alpha] = (m_\alpha - n_\alpha) L_{m_1 + n_1, \dots, m_p + n_p}^\alpha + m_\alpha d^\alpha \delta_{m_1 + n_1} \dots \delta_{m_p + n_p}.$$

The corresponding relation for the string ( $p = 1$ ) is

$$[L_m, L_n] = (m - n) \phi_{m+n}^L + (d_3 m^3 + d_1 m) \delta_{m+n}.$$

Thus one finds that in the case of the tensionless  $p$ -branes with  $p > 1$ , the central extensions in the algebra of the constraints become "smoother" since their cubic terms have to vanish due to the Jacobi identities. It is interesting to note that the

same cancellation will also occur in the case of the usual  $p$ -branes. The constraints  $L^\alpha$  are not modified by the nonzero tension and so the results obtained here for the subalgebra (3.64) are also valid for the tensile  $p$ -brane.

The investigation of the quantum theory of this model within the framework of a BRST quantization requires the introduction of new operators. To every constraint, one introduces a ghost pair  $c_{m_1, \dots, m_p}^A, b_{m_1, \dots, m_p}^A$ ,  $A \in \{-1, L_\alpha\}$ , that is fermionic. The generator of BRST transformations, the BRST charge is found to be

$$\begin{aligned}
Q = & \sum_{k_1, \dots, k_p} \phi_{-k_1, \dots, -k_p}^{-1} c_{k_1, \dots, k_p}^{-1} + \sum_{\alpha=1}^p \sum_{k_1, \dots, k_p} L^\alpha_{-k_1, \dots, -k_p} c_{k_1, \dots, k_p}^{L_\alpha} \\
& - \sum_{\alpha=1}^p \sum_{k_1, \dots, k_p} \sum_{l_1, \dots, l_p} (k_\alpha - l_\alpha) c_{-k_1, \dots, -k_p}^{-1} c_{-l_1, \dots, -l_p}^{L_\alpha} b_{k_1+l_1, \dots, k_p+l_p}^{-1} \\
& - \frac{1}{2} \sum_{\alpha, \beta=1}^p \sum_{k_1, \dots, k_p} \sum_{l_1, \dots, l_p} k_\beta c_{-k_1, \dots, -k_p}^{L_\alpha} c_{-l_1, \dots, -l_p}^{L_\beta} b_{k_1+l_1, \dots, k_p+l_p}^{L_\alpha} \\
& + \frac{1}{2} \sum_{\alpha, \beta=1}^p \sum_{k_1, \dots, k_p} \sum_{l_1, \dots, l_p} l_\alpha c_{-k_1, \dots, -k_p}^{L_\alpha} c_{-l_1, \dots, -l_p}^{L_\beta} b_{k_1+l_1, \dots, k_p+l_p}^{L_\beta}. \tag{3.65}
\end{aligned}$$

The classical nilpotency,  $Q^2 = 0$ , is guaranteed by construction. To check the nilpotency of the quantum  $\mathcal{Q} = \frac{1}{2}(Q + Q^\dagger)$ , which is constructed to be hermitian, one uses the extended constraints  $\tilde{\phi}_{m_1, \dots, m_p}^I$ . These are BRST invariant extensions of the original constraints and they satisfy the same algebra as the original ones for first rank systems. They are defined by the equation

$$\tilde{\phi}_{m_1, \dots, m_p}^I \equiv \{b_{m_1, \dots, m_p}^I, \mathcal{Q}\},$$

We can now calculate the BRST anomaly using the method described in [97]. There it is shown that

$$\mathcal{Q}^2 = \frac{1}{2} \sum_{I, J} \sum_{m_1, \dots, m_p} \tilde{d}_{m_1, \dots, m_p}^{IJ} c_{m_1, \dots, m_p}^I c_{-m_1, \dots, -m_p}^J,$$

where  $\tilde{d}^{IJ}$  are the central extensions of the extended constraints algebra. This means that

$$\mathcal{Q}^2 = \frac{1}{2} \sum_{\alpha} \tilde{d}^\alpha m_\beta c_{m_1, \dots, m_p}^{L_\alpha} c_{-m_1, \dots, -m_p}^{L_\beta}. \tag{3.66}$$

The exact values of  $\tilde{d}^\alpha$  depend on the vacuum and ordering we use. The simplest and safest method to determine these constants is to calculate the vacuum expectation value of the commutators (3.64) for the extended constraints.

According to arguments presented in [64], the vacuum suitable for tensionless strings is not one annihilated by the positive modes of the operators but one annihilated by the momenta

$$p_m^\mu |0\rangle_p = 0 \quad \forall m. \quad (3.67)$$

In the case of the tensionless  $p$ -brane, the vacuum is defined also by (3.67) in [66].

Following the prescription of [98], one takes the *ket* states to be built from our vacuum of choice,  $|0\rangle_p$ , and the *bra* states to be built from  ${}_x\langle 0|$  satisfying  ${}_x\langle 0|0\rangle_p = 1$ .

For the vacuum (3.67) and from the requirement that the BRST charge (3.65) should annihilate the vacuum, one obtains further requirements on the ghost part of the vacuum. Doing this one finds that the vacuum has to satisfy the following conditions

$$\begin{aligned} p_{m_1, \dots, m_p}^\mu |0\rangle &= b_{m_1, \dots, m_p}^{-1} |0\rangle = 0, \\ \langle 0| x_{m_1, \dots, m_p}^\mu &= \langle 0| c_{m_1, \dots, m_p}^{-1} = 0. \end{aligned}$$

The expectation value of the commutator (3.64) is

$$\begin{aligned} \langle 0| [\tilde{L}_{m_1, \dots, m_p}^\alpha, \tilde{L}_{-m_1, \dots, -m_p}^\alpha] |0\rangle &= 2m_\alpha \langle 0| \tilde{L}_{0, \dots, 0}^\alpha |0\rangle + m_\alpha \tilde{d}^\alpha \\ \Rightarrow 0 &= 2m_\alpha a_L^\alpha + m_\alpha \tilde{d}^\alpha \Rightarrow \tilde{d}^\alpha = -2a_L^\alpha \end{aligned} \quad (3.68)$$

where  $a_L^\alpha \equiv \langle 0| \tilde{L}_{0, \dots, 0}^\alpha |0\rangle$ . But for a hermitian BRST charge  $\mathcal{Q}$  one will have

$$a_L^\alpha = -\frac{1}{2}(D+1+p) \sum_{k_\alpha=-\infty}^{+\infty} k_\alpha \sum_{k_1, \dots, k_{\alpha-1}, k_{\alpha+1}, \dots, k_p} 1 = 0.$$

Thus from the relations (3.66) and (3.68) one deduces that the BRST charge is nilpotent for any space-time dimension  $D$ . So in the theory of tensionless  $p$ -branes, just as in the theory of tensionless strings the critical dimension is absent and the theory is quantum mechanically consistent for any dimension  $D$ .



If we choose the vacuum to be annihilated by the positive modes, as is the case for the usual string, the commutators (3.64) will give

$$\begin{aligned}
& \langle 0 | [\tilde{L}_{m_1, \dots, m_p}^\alpha, \tilde{L}_{-m_1, \dots, -m_p}^\alpha] | 0 \rangle = 2m_\alpha \langle 0 | \tilde{L}_{0, \dots, 0}^\alpha | 0 \rangle + m_\alpha \tilde{d}^\alpha \\
\Rightarrow & [(D - 25 - p)m_\alpha^3 - (D - 1 - p)m_\alpha] \sum_{k_1, \dots, k_{\alpha-1}, k_{\alpha+1}, \dots, k_p} \frac{1}{6} = 2m_\alpha a_L^\alpha + m_\alpha \tilde{d}^\alpha \\
\Rightarrow & D = 25 + p.
\end{aligned} \tag{3.69}$$

This results agrees with the critical dimension of  $D = 27$  for the membrane ( $p = 2$ ) which was given in [85], and can be generalized to the case of null spinning  $p$ -brane [66]

$$D = \frac{100 + 4p - 22N}{4 + N}, \tag{3.70}$$

$N$  being the number of world-volume supersymmetries.

The isomorphism  $C_{D-1,1} \simeq O(D, 2)$  for  $D \geq 3$  makes it possible to construct a theory in two extra dimensions such that the previous model corresponds to a particular gauge fixing of the latter and the conformal symmetry is manifest and linearly realized. This *conformal* tensionless  $p$ -brane action can be given by

$$S = \int d^{p+1} \xi \{ V^a (\partial_a + W_a) X^A V^b (\partial_b + W_b) X_A + \Phi X^A X_A \}, \tag{3.71}$$

where  $A = 0, \dots, d+1$  and the new metric has the form

$$\eta_{AB} = \begin{pmatrix} \eta_{\mu\nu} & 0 & 0 \\ 0 \dots 0 & 1 & 0 \\ 0 \dots 0 & 0 & -1 \end{pmatrix}.$$

$W_a$  is the gauge field for scale transformations and  $\Phi$  is a Lagrange multiplier field.

We can check that by imposing two gauge fixing conditions  $P^+ = 0$ ,  $X^+ = 1$  the generators of the Lorentz transformations in the extended space become the generators of the conformal group in the original space. Thus rotations in the extended space correspond to conformal transformations in the original space.

Going to the Hamiltonian formulation we find in exactly the same manner that the Hamiltonian is again a linear combination of the constraints. In addition to the

original constraints (3.61)-(3.62) we will have two new ones which in Fourier modes can be written as follows

$$\phi_{m_1, \dots, m_p}^1 = \frac{1}{2} \sum_{k_1, \dots, k_p = -\infty}^{+\infty} x_{m_1 - k_1, \dots, m_p - k_p} \cdot x_{k_1, \dots, k_p} = 0, \quad (3.72)$$

$$\phi_{m_1, \dots, m_p}^0 = \frac{1}{2} \sum_{k_1, \dots, k_p = -\infty}^{+\infty} p_{m_1 - k_1, \dots, m_p - k_p} \cdot x_{k_1, \dots, k_p} = 0. \quad (3.73)$$

The constraint algebra with the central extensions included, for  $p > 1$ , will be given by

$$[\phi_{m_1, \dots, m_p}^{-1}, \phi_{n_1, \dots, n_p}^1] = -2i\phi_{m_1+n_1, \dots, m_p+n_p}^0 - 2ic\delta_{m_1+n_1} \dots \delta_{m_p+n_p}, \quad (3.74)$$

$$[\phi_{m_1, \dots, m_p}^{-1}, \phi_{n_1, \dots, n_p}^0] = -i\phi_{m_1+n_1, \dots, m_p+n_p}^{-1}, \quad (3.75)$$

$$[\phi_{m_1, \dots, m_p}^{-1}, L_{n_1, \dots, n_p}^\alpha] = (m_\alpha - n_\alpha)\phi_{m_1+n_1, \dots, m_p+n_p}^{-1}, \quad (3.76)$$

$$[\phi_{m_1, \dots, m_p}^1, \phi_{n_1, \dots, n_p}^0] = i\phi_{m_1+n_1, \dots, m_p+n_p}^1, \quad (3.77)$$

$$[\phi_{m_1, \dots, m_p}^1, L_{n_1, \dots, n_p}^\alpha] = (m_\alpha + n_\alpha)\phi_{m_1+n_1, \dots, m_p+n_p}^1, \quad (3.78)$$

$$[\phi_{m_1, \dots, m_p}^0, L_{n_1, \dots, n_p}^\alpha] = m_\alpha\phi_{m_1+n_1, \dots, m_p+n_p}^0 + cm_\alpha\delta_{m_1+n_1} \dots \delta_{m_p+n_p}, \quad (3.79)$$

$$[L_{m_1, \dots, m_p}^\alpha, L_{n_1, \dots, n_p}^\beta] = m_\beta L_{m_1+n_1, \dots, m_p+n_p}^\alpha - n_\alpha L_{m_1+n_1, \dots, m_p+n_p}^\beta + \frac{1}{2} (m_\beta d^\alpha - n_\alpha d^\beta) \delta_{m_1+n_1} \dots \delta_{m_p+n_p}. \quad (3.80)$$

With the constraint algebra at hand one finds the BRST charge to be

$$\begin{aligned} Q = & \sum_{k_1, \dots, k_p} \left[ \phi_{-k_1, \dots, -k_p}^{-1} c_{k_1, \dots, k_p}^{-1} + \sum_{\alpha=1}^p L_{-k_1, \dots, -k_p}^\alpha c_{k_1, \dots, k_p}^{L_\alpha} \right. \\ & \left. + \phi_{-k_1, \dots, -k_p}^0 c_{k_1, \dots, k_p}^0 + \phi_{-k_1, \dots, -k_p}^1 c_{k_1, \dots, k_p}^1 \right] \\ & + \sum_{k_1, \dots, k_p} \sum_{l_1, \dots, l_p} \left[ 2ic_{-k_1, \dots, -k_p}^{-1} c_{-l_1, \dots, -l_p}^1 b_{k_1+l_1, \dots, k_p+l_p}^0 \right. \\ & \left. + ic_{-k_1, \dots, -k_p}^{-1} c_{-l_1, \dots, -l_p}^0 b_{k_1+l_1, \dots, k_p+l_p}^{-1} - ic_{-k_1, \dots, -k_p}^1 c_{-l_1, \dots, -l_p}^0 b_{k_1+l_1, \dots, k_p+l_p}^1 \right. \\ & \left. - \frac{1}{2} \sum_{\alpha, \beta=1}^p k_\beta c_{-k_1, \dots, -k_p}^{L_\alpha} c_{-l_1, \dots, -l_p}^{L_\beta} b_{k_1+l_1, \dots, k_p+l_p}^{L_\alpha} \right. \\ & \left. + \frac{1}{2} \sum_{\alpha, \beta=1}^p l_\alpha c_{-k_1, \dots, -k_p}^{L_\alpha} c_{-l_1, \dots, -l_p}^{L_\beta} b_{k_1+l_1, \dots, k_p+l_p}^{L_\beta} \right. \\ & \left. - \sum_{\alpha=1}^p (k_\alpha - l_\alpha) c_{-k_1, \dots, -k_p}^1 c_{-l_1, \dots, -l_p}^{L_\alpha} b_{k_1+l_1, \dots, k_p+l_p}^1 \right. \\ & \left. - \sum_{\alpha=1}^p (k_\alpha - l_\alpha) c_{-k_1, \dots, -k_p}^{-1} c_{-l_1, \dots, -l_p}^{L_\alpha} b_{k_1+l_1, \dots, k_p+l_p}^{-1} \right] \end{aligned}$$

$$- \sum_{\alpha=1}^p k_{\alpha} c_{-k_1, \dots, -k_p}^0 c_{-l_1, \dots, -l_p}^{L_{\alpha}} b_{k_1+l_1, \dots, k_p+l_p}^0 \Big]. \quad (3.81)$$

Again one can check the nilpotency of  $\mathcal{Q}$  with the use of the extended constraints. They satisfy the same algebra as the original constraints. The only thing that remains now is to calculate the values of the constants  $\tilde{d}^{\alpha}$ ,  $\alpha = 1, \dots, p$  and  $\tilde{c}$  for the vacuum and ordering introduced previously. In this case, the condition  $p_{m_1, \dots, m_p}^A |0\rangle = 0$  together with the requirement that the BRST charge (3.81) should annihilate the vacuum gives the following consistency conditions  $\forall m_1, \dots, m_p$

$$\begin{aligned} p_{m_1, \dots, m_p}^A |0\rangle &= c_{m_1, \dots, m_p}^1 |0\rangle = b_{m_1, \dots, m_p}^{-1} |0\rangle = 0, \\ \langle 0 | x_{m_1, \dots, m_p}^A &= \langle 0 | c_{m_1, \dots, m_p}^{-1} = \langle 0 | b_{m_1, \dots, m_p}^1 = 0. \end{aligned}$$

In the same way as was done previously, one finds that  $\tilde{d}^{\alpha} = 0$ . The value of  $\tilde{c}$  is found by calculating the expectation value of the commutator  $\langle 0 | [\tilde{\phi}_{m_1, \dots, m_p}^0, \tilde{\phi}_{-m_1, \dots, -m_p}^L] |0\rangle$ . This gives

$$\begin{aligned} \langle 0 | [\tilde{\phi}_{m_1, \dots, m_p}^0, \tilde{\phi}_{-m_1, \dots, -m_p}^L] |0\rangle &= m_{\alpha} \langle 0 | \tilde{\phi}_{0, \dots, 0}^0 |0\rangle + \tilde{c} m_{\alpha} \\ \Rightarrow 0 &= m_{\alpha} a_0 + \tilde{c} m_{\alpha} = 0 \\ \Rightarrow \tilde{c} &= -a_0. \end{aligned}$$

But for a Hermitian BRST charge  $\mathcal{Q}$  we will have

$$\alpha_0 \equiv \langle 0 | \tilde{\phi}_0^0 |0\rangle = -\frac{i}{4} [D - 2] \left( \sum_{k_1, \dots, k_p} 1 \right).$$

The BRST charge is nilpotent when all the constants  $\tilde{d}_a$  and  $\tilde{c}$  are equal to 0. In particular we must have

$$\tilde{c} = 0 \Rightarrow D = 2.$$

Thus one finds a critical dimension of  $D = 2$  for all the tensionless conformal  $p$ -branes. It should be stated here that the result is valid for  $p \neq 0$  since the quantum theory of the conformal particle is consistent in any dimension.

The above results can be easily generalized to the case of the conformal tensionless spinning  $p$ -brane. Using the techniques presented here, one finds for the  $p$ -brane

a negative critical dimension

$$D = 2 - 2N, \quad \forall p \geq 1,$$

$N$ , being the number of supersymmetries. Again this result is valid for  $p \neq 0$  since for the conformal spinning particle the same analysis reveals consistency in any dimension.

The first work on the classical dynamics of tensionless branes ( $p > 1$ ) in curved background is [57] (unpublished). Because of this, we will consider the results obtained in it in some detail.

The action for null  $p$ -branes in a cosmological background  $G_{MN}(x)$  may be written as

$$S = \int d^{p+1}\xi \frac{\det(\partial_\mu x^M G_{MN}(x) \partial_\nu x^N)}{E(\tau, \sigma^n)}, \quad (3.82)$$

$$(M, N = 0, \dots, D-1), \quad (m, n = 1, \dots, p),$$

where  $E(\tau, \sigma^n)$  is a  $(p+1)$ -dimensional world-volume density. The determinant  $g$  of the induced null  $p$ -brane metric  $g_{\mu\nu}$

$$g_{\mu\nu} = \partial_\mu x^M G_{MN}(x) \partial_\nu x^N = \begin{pmatrix} \dot{x}^A G_{AB}(x) \dot{x}^B & \dot{x}^A G_{AB}(x) \partial_n x^B \\ \partial_m x^A G_{AB}(x) \dot{x}^B & \hat{g}_{mn}(x) \end{pmatrix},$$

$$\hat{g}_{mn}(x) = \partial_m x^A G_{AB}(x) \partial_n x^B,$$

may be presented in a factorized form

$$g = \dot{x}^M \tilde{\Pi}_{MN}(x) \dot{x}^N \hat{g}, \quad \hat{g} = \det \hat{g}_{mn},$$

where point denotes differentiation with respect to  $\tau$ . The matrix

$$\tilde{\Pi}_{MN} = G_{MN} - G_{MB} \partial_m x^B \hat{g}^{-1mn} \partial_n x^L G_{LN}$$

has the properties of a projection operator. Therefore, the action (3.82) can be written in the following form:

$$S = \int d^{p+1}\xi \frac{\dot{x}^M \tilde{\Pi}_{MN}(x) \dot{x}^N \hat{g}}{E(\tau, \sigma^n)}.$$

The variation of this action with respect to  $E$  generates the degeneracy condition for the induced metric  $g_{\mu\nu}$

$$g = \det g_{\mu\nu} = 0,$$

which separates the class of  $(p+1)$ -dimensional isotropic geodesic hypersurfaces characterized by the null volume. In the gauge

$$\dot{x}^M G_{MN} \partial_m x^N = 0; \quad \left( \frac{\hat{g}}{E} \right)^\bullet = 0$$

one finds the equations of motion and constraints in the following form

$$\begin{aligned} \ddot{x}^M + \Gamma_{PQ}^M \dot{x}^P \dot{x}^Q &= 0 \\ \dot{x}^M G_{MN} \dot{x}^N &= 0, \quad \dot{x}^M G_{MN} \partial_m x^N = 0 \end{aligned} \quad (3.83)$$

Now consider the case of  $D$ -dimensional Friedmann universe with  $k = 0$  (flat spatial section) described by the metric form

$$ds^2 = G_{MN} dx^M dx^N = (dx^0)^2 - R^2(x^0) dx^i \delta_{ik} dx^k. \quad (3.84)$$

It is convenient to transform equations (3.83) to the conformal time  $\tilde{x}^0(\tau, \sigma^n)$ , defined by

$$dx^0 = C(\tilde{x}^0) d\tilde{x}^0, \quad C(\tilde{x}^0) = R(x^0), \quad \tilde{x}^i = x^i \quad (3.85)$$

In the gauge of conformal time, the metric (3.84) is presented in the conformal-flat form

$$ds^2 = C(\tilde{x}^0) \eta_{MN} d\tilde{x}^M d\tilde{x}^N, \quad \eta_{MN} = \text{diag}(1, -\delta_{ij}) \quad (3.86)$$

with Christoffel symbols  $\tilde{\Gamma}_{PQ}^M(\tilde{x})$

$$\tilde{\Gamma}_{PQ}^M(\tilde{x}) = C^{-1}(\tilde{x}) [\delta_P^M \tilde{\partial}_Q C + \delta_Q^M \tilde{\partial}_P C - \eta_{PQ} \tilde{\partial}^M C]. \quad (3.87)$$

Taking into account the relations (3.85), (3.87), one can transform equations (3.83) to the form

$$\ddot{\tilde{x}}^M + 2C^{-1} \dot{C} \dot{\tilde{x}}^M = 0$$

$$\eta_{MN}\dot{\tilde{x}}^M\dot{\tilde{x}}^N = 0, \quad \eta_{MN}\dot{\tilde{x}}^M\partial_m\tilde{x}^N = 0.$$

The first integration of these equations leads to the following first order equations

$$H^*C^2\dot{\tilde{x}}^0 = \psi^0(\sigma^1, \sigma^2, \dots, \sigma^p), \quad H^*C^2\dot{\tilde{x}}^i = \psi^i(\sigma^1, \sigma^2, \dots, \sigma^p), \quad (3.88)$$

the solutions of which have the form

$$\tau = H^*\psi_0^{-1} \int_{t_0}^t dt R(t), \quad (3.89)$$

$$x^i(\tau, \sigma^1, \sigma^2, \dots, \sigma^p) = H^{*-1}\psi^i \int_0^\tau d\tau R^{-2}(t),$$

where  $H^*$  is a metric constant with dimension  $L^{-1}$ ,  $t_0 \equiv x^0(0, \sigma^1, \sigma^2, \dots, \sigma^p)$ , and  $x^i(0, \sigma^1, \sigma^2, \dots, \sigma^p)$ ,  $\psi^M(\sigma^n)$  are the initial data. The solution (3.89) for the space world coordinates  $x^i(t) (i = 1, \dots, D-1)$  as a function of the cosmic time  $t = x^0$ , may be written in the equivalent form as

$$x^i(t, \sigma^1, \sigma^2, \dots, \sigma^p) = x^i(t_0, \sigma^1, \sigma^2, \dots, \sigma^p) + \nu^i(\sigma^1, \sigma^2, \dots, \sigma^p) \int_{t_0}^t dt R^{-1}(t),$$

where  $\nu^i(\sigma^n) \equiv \psi_0^{-1}\psi^i$ . The explicit form of the solutions (3.89) allows to transform the constraints (3.86) into those for the Cauchy initial data:

$$\nu^i(\sigma^m)\nu^k(\sigma^n)\delta_{ik} = 1, \quad (3.90)$$

$$\partial_m x^0(0, \sigma^m) = R(x^0(0, \sigma^m)) \nu^i(\sigma^n) \delta_{ik} \partial_m x^k(0, \sigma^m), \quad (3.91)$$

where  $\nu^k(\sigma^1, \sigma^2, \dots, \sigma^p) = \psi^i \psi_0^{-1}$ . Note that the constraints (3.90) and (3.91) produce additional constraints, which are their integrability conditions

$$\partial_m(\nu_i(\sigma^n)\partial_n x^i(0, \sigma^l)) - \partial_n(\nu_i(\sigma^n)\partial_m x^i(0, \sigma^l)) = 0.$$

Now one can show that the null p-branes may be considered as dominant gravity sources of the Friedmann universes. To this end, one assumes that the perfect fluid of these null p-branes is homogenous and isotropic. The energy density  $\rho(t)$  and the pressure  $p(t)$  of this fluid and its energy-momentum  $\langle T_{MN} \rangle$  are connected by the standard relations

$$\langle T_0^0 \rangle = \rho(t), \quad \langle T_i^j \rangle = -p(t)\delta_i^j = -\frac{\delta_i^j}{D-1} \frac{A}{R^D(t)}, \quad (3.92)$$

The tensor  $\langle T_{MN} \rangle$  is derived from the momentum-energy tensor  $T_{MN}$  of a null  $p$ -brane by means of its space averaging when a set of null  $p$ -branes is introduced instead of a separate null  $p$ -brane. The energy-momentum tensor  $T^{MN}(x)$  of null  $p$ -brane is defined by the variation of the action (3.82) with respect to  $G_{MN}(x)$

$$T^{MN}(X) = \frac{1}{\pi\gamma^*\sqrt{|G|}} \int d\tau d^p\xi \dot{x}^M \dot{x}^N \delta^D(X^M - x^M) \quad (3.93)$$

After the substitution of the velocities  $\tilde{x}^m$  (3.88) and subsequent integration with respect to  $\tau$ , the non-zero components of  $T_{MN}$  (3.93) take the following form

$$T^{00}(X) = \frac{1}{\pi\gamma^*H^*} R^{-D}(t) \int d\tau d^p\xi \psi_0(\sigma^m) \delta^{D-1}(X^i - x^i(\tau, \sigma^m)),$$

$$T^{ik}(X) = \frac{1}{\pi\gamma^*H^*} R^{-(D+2)} \int d\tau d^p\xi \{ \nu^i(\sigma) \nu^k(\sigma) \psi_0(\sigma) \delta^{D-1}(X^i - x^i(\tau, \sigma^m)) \},$$

where the time dependence in  $T^{MN}$  is factorized and accumulated in the scale factor  $R(t)$ . The constraint  $\nu^i(\sigma^m) \nu^k(\sigma^n) \delta_{ik} = 1$  gives rise to the following relation between the components of the tensor  $\langle T_{MN} \rangle$

$$Tr T = T_0^0 + G_{ij} T^{ij} = 0.$$

As a result of the space averaging, one finds the non-zero components of  $\langle T_{MN} \rangle$  to be equal to

$$\langle T_0^0 \rangle = \rho(t) = \frac{A}{R^D(t)}, \quad \langle T_i^j \rangle = -p(t) \delta_i^j, \quad (3.94)$$

where  $A$  is a constant with dimension  $L^{-D}$ . The equations (3.94) show that the equation of state of null  $p$ -branes fluid is just the equation of state for a gas of massless particles

$$\langle Tr T \rangle = \langle T_M^M \rangle = 0 \iff \rho = (D-1)p. \quad (3.95)$$

Now assume that the fluid of null  $p$ -branes is a dominant source of the Friedmann-Robertson-Walker (FRW) gravity (3.84). For the validity of the last conjecture it is necessary that the Hilbert-Einstein (HE) equations

$$R_M^N = 8\pi G_D \langle T_M^N \rangle \quad (3.96)$$

with non-zero Ricci tensor  $R_M^N$  components defined by  $G_M^N$  (3.84)

$$R_0^0 = -\frac{D-1}{R} \frac{d^2 R}{dt^2},$$

$$R_i^k = -\delta_i^k \left[ \frac{1}{R} \frac{d^2 R}{dt^2} + \frac{D-2}{R^2} \left( \frac{dR}{dt} \right)^2 \right]$$

should contain the tensor  $\langle T_M^N \rangle$  (3.92) as a source of the FRW gravity. Moreover, the constraints (3.94), i.e.

$$\rho R^D - A = 0,$$

must be an integral of motion for the HE system (3.96). It is actually realized because

$$\frac{d}{dt}(\rho R^D) = -\frac{D-2}{16\pi G_D} R^{D-1} \frac{dR}{dt} R_M^M = 0,$$

since the trace  $R_M^M \sim \langle T_M^M \rangle = 0$  (see (3.95)). In view of this fact it is enough to consider only one equation of the system (3.96)

$$\left( \frac{1}{R} \frac{dR}{dt} \right)^2 = \frac{16\pi G_D}{(D-1)(D-2)} \frac{A}{R^D} \quad (3.97)$$

which defines the scale factor  $R(t)$  of the FRW metric (3.84). Note that when  $D = 4$ , the equation (3.97) transforms into the well-known Friedmann equation for the energy density in the radiation dominated universe with  $k = 0$ . The solutions of equation (3.97) are

$$R_I(t) = [q(t_c - t)]^{2/D}, \quad t < t_c,$$

$$R_{II}(t) = [q(t - t_c)]^{2/D}, \quad t > t_c,$$

where  $q = [4\pi G_D A / (D-1)(D-2)]^{1/2}$  and  $t_c$  is a constant of integration which is a singular point of the metric. The solution  $R_I$  describes the stage of negatively accelerated contraction of  $D$ -dimensional FRW universe. The second solution  $R_{II}$  describes the stage of negatively accelerated expansion of the FRW universe from a state with space volume equal to "zero". Thus we see that the perfect fluid of non-interacting null  $p$ -branes may be considered as an alternative source of the gravity in FRW universes with  $k = 0$ .



Now let us give the Hamiltonian for null  $p$ -brane theory in curved space-time in this formulation. The canonical momentum of null  $p$ -brane  $\mathcal{P}_M$  conjugated to its world coordinate  $x^M$  is

$$\mathcal{P}_M = 2E^{-1} \hat{g} \tilde{\Pi}_{MN}(x) \dot{x}^N.$$

Then one finds the following primary constraints

$$G_{MN} \partial_m x^N \mathcal{P}^M = 0 \quad .$$

The Hamiltonian density produced by the action functional (3.82) is

$$\mathcal{H}_0 = \frac{1}{4} E \hat{g}^{-1} G^{MN} \mathcal{P}_M \mathcal{P}_N.$$

The condition for conservation of the primary constraint

$$\mathcal{P}_{(E)} = 0,$$

where  $\mathcal{P}_{(E)}$  is the canonical momentum conjugated to  $E$ , generates the following condition

$$\dot{\mathcal{P}}_{(E)} = \int d^p \xi \{ \mathcal{H}_0, \mathcal{P}_{(E)} \} = -\frac{1}{4\hat{g}} G^{MN} \mathcal{P}_M \mathcal{P}_N = 0.$$

The latter produces a secondary constraint

$$G^{MN} \mathcal{P}_M \mathcal{P}_N = 0.$$

Additional constraints do not appear, so the total Hamiltonian of null  $p$ -brane is given by

$$H = \int d^p \xi \left[ \lambda^m (G_{MN} \partial_m x^N \mathcal{P}^M) + \frac{E}{4\hat{g}} G^{MN} \mathcal{P}_M \mathcal{P}_N + \omega \mathcal{P}_{(E)} \right],$$

This hamiltonian and the reparametrization constraints may be used for the quantization of null  $p$ -branes in a curved space-time.

Finally, let us consider the interaction of null bosonic membranes ( $p = 2$ ) with antisymmetric tensor field  $T_{\mu\nu\lambda}$  in four space-time dimensions [73].

In the general case of null  $p$ -brane living in  $D$ -dimensional space, the interaction with the antisymmetric field  $T_{\mu\nu\dots\lambda}$  may be introduced in the action as follows [73]

$$S = \int d\xi^{p+1} \left[ \frac{\det(\partial_J x^\mu \partial_K x_\mu)}{2E} - \tilde{\lambda} \epsilon^{J_1 \dots J_l} \partial_{J_1} x^{\mu_1} \dots \partial_{J_l} x^{\mu_l} T_{\mu_1 \dots \mu_l} \right]. \quad (3.98)$$

In the gauge

$$\dot{x}^\nu \partial_j x_\nu = 0, \quad E = e(\sigma^j) \det(\partial_j x^\mu \partial_k x_\mu),$$

one obtains from (3.98) the equations

$$\ddot{x}_\nu + \lambda \epsilon^{J_1 \dots J_l} \partial_{J_1} x^{\mu_1} \dots \partial_{J_l} x^{\mu_l} \partial_{[\nu} T_{\mu_1 \dots \mu_l]} = 0, \quad \dot{x}^2 = 0,$$

where

$$\partial_{[\nu} T_{\mu_1 \dots \mu_l]} \equiv \partial_\nu T_{\mu_1 \dots \mu_l} - \partial_{\mu_1} T_{\nu \mu_2 \dots \mu_l} - \dots - \partial_{\mu_l} T_{\mu_1 \dots \mu_{l-1} \nu} \neq 0.$$

From now on, we restrict ourselves to the particular case  $D = 4$ ,  $p = 2$ . The world vector  $x^\mu$  of the null membrane interacting with the field  $T_{\lambda\mu\nu}$  which is dual to the vector field  $T_\kappa$  ( $T_{\lambda\mu\nu} = T^\kappa \epsilon_{\kappa\lambda\mu\nu}$ ) is defined as

$$\dot{x}^\mu \dot{x}_\mu = 0, \quad \dot{x}^\mu \partial_j x_\mu = 0, \quad (j = 1, 2), \quad (3.99)$$

$$\ddot{x}_\kappa + \lambda \epsilon_{\kappa\mu\nu\lambda} \partial_J x^\mu \partial_K x^\nu \partial_L x^\lambda \epsilon^{JKL} \partial_\rho T^\rho(x) = 0, \quad \lambda = e(\sigma^j) \tilde{\lambda}. \quad (3.100)$$

At first, we consider equation (3.99) and using the isotropic character of  $\dot{x}^\nu$  present this vector, as well as the vectors  $\partial_j x^\nu$  which are orthogonal to  $\dot{x}^\nu$ , in the spinor form

$$\begin{aligned} \dot{x}_{A\dot{A}} &= u_A \bar{u}_{\dot{A}}, \\ \partial_1 x_{A\dot{A}} &= u_A \bar{v}_{\dot{A}} + v_A \bar{u}_{\dot{A}}, \\ \partial_2 x_{A\dot{A}} &= u_A \bar{w}_{\dot{A}} + w_A \bar{u}_{\dot{A}}. \end{aligned} \quad (3.101)$$

The spinor basis used in (3.101) is defined as

$$\partial_J x^\mu = -\frac{1}{2} (\tilde{\sigma}^\mu)^{\dot{A}A} \partial_J x_{A\dot{A}}, \quad \partial_J x_{A\dot{A}} = \sigma_{A\dot{A}}^\mu \partial_J x_\mu.$$

In the spinor basis, it is evident that the representation (3.101) is the general solution of (3.99), because

$$\begin{aligned} u^A u_A = 0 = \bar{u}_{\dot{A}} \bar{u}^{\dot{A}}, \quad u^A = \epsilon^{AB} u_B, \quad \bar{u}^{\dot{A}} = \epsilon^{\dot{A}\dot{B}} \bar{u}_{\dot{B}}, \\ \epsilon^{AB} = -\epsilon^{BA}, \quad \epsilon^{12} = 1. \end{aligned}$$

After solving two of three integrability conditions  $(\partial_j x^\nu)^\cdot = \partial_j \dot{x}^\nu$ , one finds the following representations for the spinors  $u^A$ ,  $v^A$ ,  $w^A$ :

$$\begin{aligned} u_A &= \exp[r(\tau, \sigma^j)] \alpha_A(\sigma^j), \quad \bar{u}_{\dot{A}} = \exp[\bar{r}(\tau, \sigma^j)] \bar{\alpha}_{\dot{A}}(\sigma^j), \\ v_A &= \exp(-\bar{r}) \left\{ \beta_A(\sigma^j) + \int_0^\tau d\tilde{\tau} [\partial_1 u_A + i\mu(\tilde{\tau}, \sigma^j) u_A] \exp(\bar{r}) \right\}, \\ w_A &= \exp(-\bar{r}) \left\{ \gamma_A(\sigma^j) + \int_0^\tau d\tilde{\tau} [\partial_2 u_A + i\nu(\tilde{\tau}, \sigma^j) u_A] \exp(\bar{r}) \right\}. \end{aligned} \quad (3.102)$$

Using (3.102), we conclude that the vectors  $\partial_j x_{A\dot{A}}$  constrained by the equation (3.99) can be presented in the form

$$\begin{aligned} \dot{x}_{A\dot{A}} &= R(\tau, \sigma^j) \alpha_A \bar{\alpha}_{\dot{A}}, \quad R \equiv \int_0^\tau d\tilde{\tau} \exp(r + \bar{r}), \\ \partial_1 x_{A\dot{A}} &= (\alpha_A \bar{\beta}_{\dot{A}} + \beta_A \bar{\alpha}_{\dot{A}}) + \partial_1 (R \alpha_A \bar{\alpha}_{\dot{A}}), \\ \partial_2 x_{A\dot{A}} &= (\alpha_A \bar{\gamma}_{\dot{A}} + \gamma_A \bar{\alpha}_{\dot{A}}) + \partial_2 (R \alpha_A \bar{\alpha}_{\dot{A}}). \end{aligned} \quad (3.103)$$

The last integrability condition  $\partial_1(\partial_2 x^\nu) = \partial_2(\partial_1 x^\nu)$  may be written in the form

$$\partial_1 (\alpha_A \bar{\gamma}_{\dot{A}} + \gamma_A \bar{\alpha}_{\dot{A}}) = \partial_2 (\alpha_A \bar{\beta}_{\dot{A}} + \beta_A \bar{\alpha}_{\dot{A}}). \quad (3.104)$$

In particular, equation (3.104) has the solution  $\beta_A = \partial_1 \alpha_A$ ,  $\gamma_A = \partial_2 \alpha_A$ , which generates the following world vector

$$x^\mu(\tau, \sigma^j) = [1 + R(\tau, \sigma^j)] (\bar{\alpha} \sigma^\mu \alpha) + q^\mu. \quad (3.105)$$

If we introduce the vectors  $a^\mu = -\frac{1}{2} (\bar{\alpha} \sigma^\mu \alpha)$  and  $b^\mu(\sigma^j)$  composed of the spinors  $\alpha_A$ ,  $\beta_A$ ,  $\gamma_A$  and satisfying the following relations

$$a^\mu a_\mu = 0, \quad a^\nu \partial_j b_\nu = 0, \quad (3.106)$$

the general solution of (3.99) may be written in the form

$$x^\mu(\tau, \sigma^j) = R(\tau, \sigma^j) a^\mu(\sigma^j) + b^\mu(\sigma^j). \quad (3.107)$$

Taking into account the invariance of (3.99) under world-volume reparametrizations defined by the relations  $\tau \rightarrow \tilde{\tau}(\tau, \sigma^j)$ ,  $\sigma^j \rightarrow \tilde{\sigma}^j(\sigma^k)$ , one can choose  $R$  as a new proper time:  $\tilde{\tau} = R$ .

In the new coordinates  $(R, \sigma^j)$ , the general solution (3.107) takes the form of the general solution of free null membrane equations of motion. Hence, the null membrane interactions with the  $T_{\lambda\mu\nu}$  field can be compensated by the reparametrizations of its world-volume. The  $R$ -function is defined by the left hand side of (3.100), which may be written in the following form

$$\ddot{x}_\mu - 6\lambda\Omega_\mu\partial_\nu T^\nu = 0,$$

where  $\Omega_\mu$  is the four-vector of an element of the light-like volume

$$\Omega^\lambda = -\frac{1}{8}\dot{x}_{A\dot{A}}\partial_1 x_{B\dot{B}}\partial_2 x_{C\dot{C}}\epsilon^{\lambda\mu\nu\rho}\tilde{\sigma}_\mu^{\dot{A}A}\tilde{\sigma}_\nu^{\dot{B}B}\tilde{\sigma}_\rho^{\dot{C}C},$$

written in the spinor basis.  $\Omega_\mu$  can be presented also in the form

$$\Omega^\lambda = \frac{i}{4}\left[\epsilon^{AB}\epsilon^{\dot{A}\dot{C}}\left(\tilde{\sigma}^\lambda\right)^{\dot{B}C} - \epsilon^{AC}\epsilon^{\dot{A}\dot{B}}\left(\tilde{\sigma}^\lambda\right)^{\dot{C}B}\right]\dot{x}_{A\dot{A}}\partial_1 x_{B\dot{B}}\partial_2 x_{C\dot{C}}. \quad (3.108)$$

When equations (3.103) for  $\partial_J x_{A\dot{A}}$  are substituted into (3.108), one finds that the four-vector  $\Omega^\lambda$  is collinear to the light-like vector  $a^\lambda(\sigma^j)$

$$\Omega^\lambda = -\frac{1}{2}\dot{R}\left(AR^2 + BR + D\right)a^\lambda(\sigma^j), \quad (3.109)$$

where the coefficients  $A(\sigma^j)$ ,  $B(\sigma^j)$  and  $D(\sigma^j)$  are defined as

$$\begin{aligned} A(\sigma^j) &= i[(\alpha \cdot \beta)(\bar{\alpha} \cdot \bar{\gamma}) - c.c.], \\ B(\sigma^j) &= i[(\alpha \cdot \partial_1 \alpha)(\bar{\alpha} \cdot \bar{\gamma}) - (\bar{\alpha} \cdot \partial_2 \bar{\alpha})(\alpha \cdot \beta) - c.c.], \\ D(\sigma^j) &= i[(\alpha \cdot \partial_1 \alpha)(\bar{\alpha} \cdot \partial_2 \bar{\alpha}) - c.c.]. \end{aligned}$$

Taking into account the equations (3.103)-(3.107) and (3.109), one finds that (3.100) is reduced to the single equation for the function  $R$

$$\ddot{R} + 3\lambda\dot{R}\left(AR^2 + BR + D\right)\partial_\nu T^\nu(R, a^\mu, b^\rho) = 0,$$

which can be integrated. Its general solution may be presented as

$$\begin{aligned}\tau &= \int \frac{dR}{f(R)} + \tilde{C}_2(\sigma^j), \\ f(R) &= -\lambda \int dR \left( AR^2 + BR + D \right) \partial_\nu T^\nu(R, a^\mu, b^\rho) + \tilde{C}_1(\sigma^j).\end{aligned}$$

In the particular case when  $T_{\mu\nu\lambda} = x^\rho \epsilon_{\rho\mu\nu\lambda}$ , this solution is expressed by the table integral

$$\lambda\tau + C_2(\sigma^j) = -\frac{1}{4} \int \frac{dR}{\frac{1}{2}AR^3 + \frac{1}{2}BR^2 + DR + C_1(\sigma^j)}. \quad (3.110)$$

The relation (3.110) establishes a connection between the old proper time  $\tau$  and the new one  $R$ . The usage of the new world-volume coordinates  $\tilde{\tau} = R$ ,  $\sigma^j$  leads to the exclusion of the field  $T_{\lambda\mu\nu}$  from equations (3.99), (3.100) in the considered particular case.

## 4 TENSIONLESS BRANES AND THE NULL STRING CRITICAL DIMENSION

In this section we perform BRST quantization of the null bosonic  $p$ -branes using different types of operator ordering. It is shown that one can or can not obtain critical dimension for the null string ( $p = 1$ ), depending on the choice of the operator ordering and corresponding vacuum states. When  $p > 1$ , operator orderings leading to critical dimension in the  $p = 1$  case are not allowed. Admissible orderings give no restrictions on the dimension of the embedding space-time. The results described here are obtained in the paper [71].

### 4.1 Classical theory

To begin with, we first write down the Hamiltonian for the null  $p$ -branes living in  $D$ -dimensional Minkowski space-time. It can be cast in the form:

$$H = \int d^p \sigma \left( \mu^0 \varphi_0 + \mu^j \varphi_j \right),$$

where  $\mu^0, \mu^j$  are Lagrange multipliers being arbitrary functions of the time parameter  $\tau$  and volume coordinates  $\sigma^1, \dots, \sigma^p$ . The constraints  $\varphi_0, \varphi_j$  are defined by the equalities:

$$\varphi_0 = p^\mu p_\mu, \quad \varphi_j = \eta_{\mu\nu} p^\mu \partial_j x^\nu.$$

Here  $x^\mu$  and  $p_\mu$  are canonically conjugated coordinates and momenta and  $\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$ .

$\varphi_0$  and  $\varphi_j$  obey the Poisson bracket algebra

$$\begin{aligned} \{\varphi_0(\underline{\sigma}_1), \varphi_0(\underline{\sigma}_2)\} &= 0, \\ \{\varphi_0(\underline{\sigma}_1), \varphi_j(\underline{\sigma}_2)\} &= [\varphi_0(\underline{\sigma}_1) + \varphi_0(\underline{\sigma}_2)] \partial_j \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\ \{\varphi_j(\underline{\sigma}_1), \varphi_k(\underline{\sigma}_2)\} &= [\delta_j^l \varphi_k(\underline{\sigma}_1) + \delta_k^l \varphi_j(\underline{\sigma}_2)] \partial_l \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \end{aligned} \tag{4.111}$$

which means, that they are first class quantities. The Dirac consistency conditions [108]

$$\{\varphi_0, H\} \approx 0 \quad , \quad \{\varphi_j, H\} \approx 0,$$

do not place any restrictions on the Lagrange multipliers  $\mu^0, \mu^j$ .

Following the BFV-BRST method for quantization of constrained systems [79, 80, 81, 82], we now introduce for each constraint  $\varphi_0, \varphi_j$  a pair of anticommuting ghost variables  $(\eta^0, P_0), (\eta^j, P_j)$  respectively, which are canonically conjugated. Then the BRST charge is

$$Q = \int d^p \sigma \{ \varphi_0 \eta^0 + \varphi_j \eta^j + P_0 [(\partial_j \eta^j) \eta^0 + (\partial_j \eta^0) \eta^j] + P_k (\partial_j \eta^k) \eta^j \}$$

and it has the property

$$\{Q, Q\}_{pb} = 0$$

where  $\{., .\}_{pb}$  is the Poisson bracket in the extended phase space  $(x^\nu, p_\mu; \eta^0, P_0; \eta^j, P_j)$ .

In the new phase space, the constraints are given by the following brackets:

$$\begin{aligned} \varphi_0^{tot} &= \{Q, P_0\}_{pb} = \varphi_0 + 2P_0 \partial_j \eta^j + (\partial_j P_0) \eta^j = \varphi_0 + \varphi_0^{gh}, \\ \varphi_j^{tot} &= \{Q, P_j\}_{pb} = \varphi_j + 2P_0 (\partial_j \eta^0) + (\partial_j P_0) \eta^0 + P_j \partial_k \eta^k + P_k (\partial_j \eta^k) + (\partial_k P_j) \eta^k \\ &= \varphi_j + \varphi_j^{gh} \end{aligned}$$

and they are first class. The BRST invariant Hamiltonian is

$$H_\chi = \{Q, \chi\}_{pb} \quad , \quad \{Q, H_\chi\}_{pb} = 0,$$

where  $\chi$  is arbitrary, anticommuting, gauge fixing function. We choose

$$\chi = \Lambda^0 \int d^p \sigma P_0 + \Lambda^j \int d^p \sigma P_j, \quad \Lambda^0, \Lambda^j - const$$

and obtain:

$$H_\chi = \int d^p \sigma [\Lambda^0 \varphi_0^{tot} + \Lambda^j \varphi_j^{tot}]. \quad (4.112)$$

Let us note that additional set of canonically conjugated ghosts  $(\bar{\eta}_0, \bar{P}^0), (\bar{\eta}_j, \bar{P}^j)$  must be added if we wish to write down the corresponding BRST invariant Lagrangian. If so,  $Q$  and  $\chi$  have to be modified as follows

$$\tilde{Q} = Q + \int d^p \sigma (M_0 \bar{P}^0 + M_j \bar{P}^j),$$

$$\tilde{\chi} = \chi + \int d^p \sigma \left[ \bar{\eta}_0 (\chi^0 + \frac{\rho_0}{2} M^0) + \bar{\eta}_j (\chi^j + \frac{\rho_{(j)}}{2} M^j) \right],$$

where  $M_0, M_j$  are the momenta, canonically conjugated to  $\mu^0$  and  $\mu^j$  respectively,  $\chi^0$  and  $\chi^j$  are gauge fixing conditions [99, 100] for  $\varphi_0$  and  $\varphi_j$ ,  $\rho_0$  and  $\rho_{(j)}$  are parameters. All the above results in the Lagrangian density ( $\partial_\tau = \partial/\partial\tau$ ):

$$L_{\tilde{\chi}} = L + L_{GF} + L_{GH},$$

where

$$L = (1/4\mu^0)(\partial_\tau x - \mu^j \partial_j x)^2,$$

the gauge fixing part is

$$L_{GF} = \frac{1}{2\rho_0}(\partial_\tau \mu^0 - \chi^0)(\partial_\tau \mu_0 - \chi_0) + \frac{1}{2\rho_{(j)}}(\partial_\tau \mu^j - \chi^j)(\partial_\tau \mu_j - \chi_j)$$

and the ghost part is

$$\begin{aligned} L_{GH} = & -\partial_\tau \bar{\eta}_0 \partial_\tau \eta^0 - \partial_\tau \bar{\eta}_j \partial_\tau \eta^j + \mu^0 [2\partial_\tau \bar{\eta}_0 \partial_j \eta^j + (\partial_j \partial_\tau \bar{\eta}_0) \eta^j] \\ & + \mu^j [2\partial_\tau \bar{\eta}_0 \partial_j \eta^0 + (\partial_j \partial_\tau \bar{\eta}_0) \eta^0 + \partial_\tau \bar{\eta}_k \partial_j \eta^k + \partial_\tau \bar{\eta}_j \partial_k \eta^k + (\partial_k \partial_\tau \bar{\eta}_j) \eta^k] \\ & + \int d^p \sigma' \{ \bar{\eta}_0(\underline{\sigma}') [\{ \varphi_0, \chi^0(\underline{\sigma}') \}_{pb} \eta^0 + \{ \varphi_j, \chi^0(\underline{\sigma}') \}_{pb} \eta^j] \\ & + \bar{\eta}_j(\underline{\sigma}') [\{ \varphi_0, \chi^j(\underline{\sigma}') \}_{pb} \eta^0 + \{ \varphi_k, \chi^j(\underline{\sigma}') \}_{pb} \eta^k] \}. \end{aligned}$$

Let us now go back to the Hamiltonian picture. The Hamiltonian (4.112) leads to equations of motion with the following general solution for the bosonic variables

$$x^\nu = y^\nu(\underline{z}) + 2g(\tau)p^\nu(\underline{z}), \quad p_\nu = p_\nu(\underline{z}),$$

and for the ghosts [101]

$$\begin{aligned} \eta^0 &= \zeta^0(\underline{z}) + g(\tau) \partial_j \eta^j(\underline{z}), \\ P_0 &= P_0(\underline{z}), \quad \eta^j = \eta^j(\underline{z}), \\ P_j &= \Pi_j(\underline{z}) + g(\tau) \partial_j P_0(\underline{z}). \end{aligned} \tag{4.113}$$

Here  $y^\nu, p_\nu, \zeta^0, P_0, \eta^j$  and  $\Pi_j$  are arbitrary functions of the variables  $z^j$ ,

$$z^j = \Lambda^j \tau + \sigma^j \quad \text{and} \quad g(\tau) = \Lambda^0 \tau.$$



On the solutions (4.113) the BRST charge  $Q$  takes the form [101]

$$Q^S = \int d^p z \{ \phi_0 \zeta^0 + \phi_j \eta^j + P_0 [(\partial_j \eta^j) \zeta^0 + (\partial_j \zeta^0) \eta^j] + \Pi_k (\partial_j \eta^k) \eta^j \},$$

where  $\phi_0 = p^2(\underline{z})$ ,  $\phi_j = p_\nu(\underline{z}) \partial_j y^\nu(\underline{z})$ . Now the constraints are

$$\phi_0^{tot}(\underline{z}) = \{Q^S, P_0(\underline{z})\}_{pb}, \quad \phi_j^{tot}(\underline{z}) = \{Q^S, \Pi_j(\underline{z})\}_{pb},$$

and they are connected with  $\varphi_0^{tot}, \varphi_j^{tot}$  by the equalities

$$\varphi_0^{tot}(\underline{z}) = \phi_0^{tot}(\underline{z}), \quad \varphi_j^{tot} = \phi_j^{tot}(\underline{z}) + g(\tau) \partial_j \phi_0^{tot}(\underline{z}).$$

From now on, we confine ourselves to the case of periodic boundary conditions when our phase-space variables admit Fourier series expansions. Let us denote the Fourier components of  $y^\nu, p^\nu, \zeta^0, P_0, \eta^j$  and  $\Pi_j$  with  $x_{\underline{k}}^\nu, p_{\underline{k}}^\nu, c_{\underline{k}}, b_{\underline{k}}, \bar{c}_{\underline{k}}^j$  and  $\bar{b}_{j,\underline{k}}$  respectively. For the zero modes of  $p^\nu$  and  $x^\nu$ , we introduce the notations

$$P^\mu = (2\pi)^p p_{\underline{0}}^\mu, \quad q^\nu = \frac{-i}{(2\pi)^p} x_{\underline{0}}^\nu.$$

Then we have the following non-zero Poisson brackets:

$$\begin{aligned} \{P^\mu, q^\nu\}_{pb} &= -\eta^{\mu\nu}, & \{p_{\underline{k}}^\mu, x_{\underline{n}}^\nu\}_{pb} &= -i\eta^{\mu\nu} \delta_{\underline{k}+\underline{n}, \underline{0}}, \\ \{c_{\underline{k}}, b_{\underline{n}}\}_{pb} &= -i\delta_{\underline{k}+\underline{n}, \underline{0}}, & \{\bar{c}_{\underline{k}}^j, \bar{b}_{j,\underline{n}}\}_{pb} &= -i\delta_{\underline{k}}^j \delta_{\underline{k}+\underline{n}, \underline{0}}. \end{aligned} \quad (4.114)$$

The Fourier expansions for the constraints  $\phi_0^{tot}$  and  $\phi_j^{tot}$  are

$$\phi_0^{tot}(\underline{z}) = \frac{1}{(2\pi)^p} \sum_{\underline{m} \in Z^p} C_{\underline{m}}^{tot} e^{-i\underline{m}z}, \quad \phi_j^{tot}(\underline{z}) = \frac{1}{(2\pi)^p} \sum_{\underline{m} \in Z^p} D_{j,\underline{m}}^{tot} e^{-i\underline{m}z}.$$

Here

$$C_{\underline{n}}^{tot} = i\{Q^S, b_{\underline{n}}\}_{pb} = C_{\underline{n}} + C_{\underline{n}}^{gh}, \quad D_{j,\underline{n}}^{tot} = i\{Q^S, \bar{b}_{j,\underline{n}}\}_{pb} = D_{j,\underline{n}} + D_{j,\underline{n}}^{gh}, \quad (4.115)$$

where

$$\begin{aligned} Q^S &= \sum_{\underline{n} \in Z^p} \{ [C_{\underline{n}} + (1/2)C_{\underline{n}}^{gh}] c_{-\underline{n}} + [D_{j,\underline{n}} + (1/2)D_{j,\underline{n}}^{gh}] \bar{c}_{-\underline{n}}^j \}, \\ C_{\underline{n}} &= (2\pi)^p \sum_{\underline{k} \in Z^p} p_{\underline{k}}^\nu p_{\nu, \underline{n}-\underline{k}}, \\ D_{j,\underline{n}} &= - \sum_{\underline{k} \in Z^p} (n_j - k_j) p_{\underline{k}}^\nu x_{\nu, \underline{n}-\underline{k}}, \\ C_{\underline{n}}^{gh} &= \sum_{\underline{k} \in Z^p} (n_j - k_j) b_{\underline{n}+\underline{k}} \bar{c}_{-\underline{k}}^j, \\ D_{j,\underline{n}}^{gh} &= \sum_{\underline{k} \in Z^p} [(n_j - k_j) b_{\underline{n}+\underline{k}} c_{-\underline{k}} + (\delta_j^l n_k - \delta_k^l k_j) \bar{b}_{l, \underline{n}+\underline{k}} \bar{c}_{-\underline{k}}^k] \end{aligned} \quad (4.116)$$

Using expressions (4.114)-(4.116), one obtains the algebra of the total generators (4.115)

$$\begin{aligned}\{C_{\underline{n}}^{tot}, C_{\underline{m}}^{tot}\}_{pb} &= 0, \\ \{C_{\underline{n}}^{tot}, D_{j,\underline{m}}^{tot}\}_{pb} &= -i(n_j - m_j)C_{\underline{n}+\underline{m}}^{tot}, \\ \{D_{j,\underline{n}}^{tot}, D_{k,\underline{m}}^{tot}\}_{pb} &= -i(\delta_j^l n_k - \delta_k^l m_j)D_{l,\underline{n}+\underline{m}}^{tot}.\end{aligned}\tag{4.117}$$

## 4.2 Quantization

Going to the quantum theory according to the rule  $i\{.,.\}_{pb} \rightarrow$  (anti)commutator, we define  $Q^S$  by introducing the renormalized operators ( $\alpha, \beta_j$  are constants)

$$C_{\underline{n}}^{tot} = C_{\underline{n}} + C_{\underline{n}}^{gh} - \alpha\delta_{\underline{n},\underline{0}} \quad , \quad D_{j,\underline{n}}^{tot} = D_{j,\underline{n}} + D_{j,\underline{n}}^{gh} - \beta_j\delta_{\underline{n},\underline{0}} \tag{4.118}$$

and postulating [101]

$$\begin{aligned}Q^S = \sum_{\underline{n} \in Z^p} : \{ [C_{\underline{n}} + (1/2)C_{\underline{n}}^{gh} - \alpha\delta_{\underline{n},\underline{0}}]c_{-\underline{n}} \\ + [D_{j,\underline{n}} + (1/2)D_{j,\underline{n}}^{gh} - \beta_j\delta_{\underline{n},\underline{0}}]\bar{c}_{-\underline{n}}^j \} :, \end{aligned}$$

where  $:...:$  represents operator ordering and operator ordering in  $C_{\underline{n}}, \dots, D_{j,\underline{n}}^{gh}$  is also assumed.

Let us turn to the question about the critical dimensions which might appear in the model under consideration. As is well known, the critical dimension arises as a necessary condition for nilpotency of the BRST charge operator. In turn, this is connected with the vanishing of the central charges in the quantum constraint algebra. Because of that, we are going to find out the central terms which appear in our quantum gauge algebra for different values of  $p$  (the most general form of central extension, which is compatible with the Jacobi identities is written in the Appendix).

We start with the case  $p = 1$ , which corresponds to a closed string. In this case  $j = k = 1$  and one defines the operator ordering with respect to  $p_{-n}^\nu, \dots, \bar{c}_{-n}$  and  $p_n^\nu, \dots, \bar{c}_n$  ( $n > 0$ ), so that

$$p_{-n}^\nu | 0 \rangle = \dots = \bar{c}_{-n} | 0 \rangle = 0, \quad \langle 0 | p_n^\nu = \dots = \langle 0 | \bar{c}_n = 0.$$

We call this ordering "*string-like*". Using the explicit expressions for the constraints (4.116), one obtains that central terms appear in the commutators  $[D_n, D_m]$ ,  $[D_n^{gh}, D_m^{gh}]$  and they are respectively

$$c = (D/6)(n^2 - 1)n\delta_{n+m,0} \quad , \quad c^{gh} = -(1/3)(13n^2 - 1)n\delta_{n+m,0}.$$

Therefore, the quantum constraint algebra has the form

$$\begin{aligned} [C_n^{tot}, C_m^{tot}] &= 0, \\ [C_n^{tot}, D_m^{tot}] &= (n - m)C_{n+m}^{tot} + 2\alpha n\delta_{n+m,0}, \\ [D_n^{tot}, D_m^{tot}] &= (n - m)D_{n+m}^{tot} + (1/6)[(D - 26)n^2 + (12\beta - D + 2)]n\delta_{n+m,0}. \end{aligned}$$

This means that the conditions for the nilpotency of the BRST charge operator  $Q^S$  are

$$(D - 26)n^2 + (12\beta - D + 2) = 0 \quad , \quad \alpha = 0,$$

which leads to the well known result  $D = 26, \beta = 2$ . Obviously, this reproduces one of the basic features of the quantized tensionful closed bosonic string - its critical dimension.

Going to the case  $p > 1$ , one natural generalization of the creation and annihilation operators definition is

$$p_{\underline{n}}^\nu | 0 \rangle_j = 0, \quad j < 0 \mid p_{-\underline{n}}^\nu = 0, \quad \text{for} \quad \sum_{j=1}^p n_a > 0$$

and analogously for the operators  $x_{\underline{n}}^\nu, \dots, \bar{c}_{\underline{n}}^j$ . However, it turns out that such definition does not agree with the Jacobi identities for the quantum constraint algebra (except for  $p = 1$ ). That is why, we introduce the creation (+) and annihilation (-) operators as follows [101]

$$p_{\underline{n}}^\nu = (1/\sqrt{2})(p_{\underline{n}}^{\nu+} + p_{-\underline{n}}^{\nu-}), \dots, \bar{c}_{\underline{n}}^j = (1/\sqrt{2})(\bar{c}_{\underline{n}}^{j+} + \bar{c}_{-\underline{n}}^{j-}) \quad (4.119)$$

and respectively new vacuum states

$$p_{\underline{n}}^{\nu-} | vac \rangle = \dots = \bar{c}_{\underline{n}}^{j-} | vac \rangle = 0 \quad , \quad \langle vac | p_{\underline{n}}^{\nu+} = \dots = \langle vac | \bar{c}_{\underline{n}}^{j+} = 0.$$

This choice of the creation and annihilation operators corresponds to the representation of all phase-space variables  $p^\nu, \dots, \bar{c}^j$  as sums of frequency parts which are conjugated to each other and satisfy the same equation of motion as the corresponding dynamical variable.

By direct computation one shows that with operator product defined with respect to the introduced creation and annihilation operators (4.119) (we shall refer to as "*normal ordering*"), a central extension of the algebra of the gauge generators (4.118) does not appear, i.e.  $\alpha = 0, \beta_j = 0$ . Consequently, the BRST charge operator  $Q^S$  is automatically nilpotent in this case and there is no restriction on the dimension of the background space-time for  $p > 1$ .

The impossibility to introduce a *string-like* operator ordering when  $p > 1$  leads to the problem of finding those operator orderings which are possible for  $p = 1$  as well as for  $p > 1$ . First of all, we check the consistency of the (already used for  $p > 1$ ) *normal ordering* for  $p = 1$ . It turns out to be consistent, but now critical dimension for the null string does not appear. The same result - absence of critical dimension for every value of  $p$ , one obtains when uses the so called *particle-like* operator ordering. Now the *ket* vacuum is annihilated by momentum-type operators and the *bra* vacuum is annihilated by coordinate-type ones:

$$\begin{aligned} p_{\underline{n}}^\mu | 0 \rangle_M &= b_{\underline{n}} | 0 \rangle_M = \bar{b}_{\underline{n}} | 0 \rangle_M = 0, \\ C < 0 | x_{\underline{n}}^\mu &= C < 0 | c_{\underline{n}} = C < 0 | \bar{c}_{\underline{n}} = 0 \quad , \quad \forall \underline{n} \in Z^p. \end{aligned}$$

Further, we check the case when *Weyl ordering* is applied. Now it turns out, that in the null string case ( $p = 1$ ) this leads to critical dimension  $D = 26$ , but for the null brane ( $p > 1$ ) this ordering is inconsistent, as was the *string-like* one.

As a final result, we checked four types of operator orderings. Two of them are valid for the string as well as for the brane and then we do not receive any critical dimension. The other two types of orderings give critical dimension  $D = 26$  for the string and are not applicable for the brane. Our opinion is that the right operator ordering is the one applicable for all  $p = 1, 2, \dots$ . In other words, our viewpoint is that neither null strings nor null branes have critical dimensions. The same point

of view is presented in [66].

Let us discuss in more details the impossibility to introduce at  $p > 1$  an operator ordering which at  $p = 1$  gives critical dimension. This is connected with the fact that the constraint algebra, as is shown in the Appendix A, does not possess non-trivial central extension when  $p > 1$  (see also [102], [66]). As a matter of fact, the string critical dimension appears in front of  $n^3$ , i.e. in the non-trivial part of the constraint algebra central extension, which can not be taken away by simply redefining the generators  $D_n$ , in contrast to the trivial part  $\sim n$ . Because of the nonexistence of non-trivial central extension when  $p > 1$ , any critical dimension arising is impossible in view of the Jacobi identities. Therefore, if the quantum null brane constraint algebra is given by (up to trivial central extensions)

$$\begin{aligned} [C_{\underline{n}}^{tot}, C_{\underline{m}}^{tot}] &= 0, \\ [C_{\underline{n}}^{tot}, D_{j,\underline{m}}^{tot}] &= (n_j - m_j) C_{\underline{n}+\underline{m}}^{tot}, \\ [D_{j,\underline{n}}^{tot}, D_{k,\underline{m}}^{tot}] &= (\delta_j^l n_k - \delta_k^l m_j) D_{l,\underline{n}+\underline{m}}^{tot}, \end{aligned}$$

then the latter has no critical dimension and exists in any D-dimensional space-time, when embedding of the  $p + 1$ - dimensional world-volume of the  $p$ -brane is possible.

Finally, note that in each of the  $p$  subalgebras (at fixed  $j$ ) of the constraint algebra, one can obtain non-trivial central extension and consequently - critical dimension (see Appendix A). For example, taking *string-like* or *Weyl ordering*, one derives  $D = 25 + p$ . The same critical dimensions are obtained also in [66]. However, the considered quantum dynamical system is described by the *full* constraint algebra, where only trivial central extensions are possible.

### 4.3 Comments

The observation, that there is an operator ordering which is valid  $\forall p \in Z_+$  and another one, which is admissible only for  $p = 1$  [101], leads to the problem of finding those orderings which are possible for every positive integer value of  $p$ . We applied here four types of operator orderings and we establish that two of them (*normal* and *particle-like orderings*) are admissible  $\forall p \in Z_+$ , but the other two

(*string – like* and *Weyl orderings*) are admissible only for  $p = 1$ . The fact that the latter two orderings lead to the appearance of critical dimension, and the former two do not, is a consequence of the constraint algebra property to have non-trivial central extension only for  $p = 1$ . On the other hand, the obtained nontrivial central extensions of the Virasoro type for some of its subalgebras, provide an explanation why the critical dimensions  $D = 25 + p, p = 1, 2, \dots$  [66], re-derived here, can emerge. However, our claim is that the critical dimensions appearing in the subalgebras, must not be considered as such for the given model as a whole. The model is represented by the full constraint algebra, which does not possess non-trivial central extension for  $p \geq 2$ . This leads us to the proposition of the rule: the right operator orderings in the case of null string ( $p = 1$ ) are those which are also admissible in the  $p > 1$  case.

## 5 NULL BRANES IN CURVED BACKGROUNDS

In this section, we consider null bosonic  $p$ -branes moving in curved space-times. Some exact solutions of the classical equations of motion and of the constraints for the null membrane in general stationary, axially symmetric, four dimensional gravity background are found. The results considered here are obtained in [74].

### 5.1 Classical formulation

The action for the bosonic null  $p$ -brane in a  $D$ -dimensional curved space-time with metric tensor  $g_{\mu\nu}(x)$  can be written in the form:

$$S = \int d^{p+1} \xi L, \quad L = V^J V^K \partial_J x^\mu \partial_K x^\nu g_{\mu\nu}(x), \quad (5.120)$$

$$\partial_J = \partial / \partial \xi^J, \quad \xi^J = (\xi^0, \xi^j) = (\tau, \sigma^j),$$

$$J, K = 0, 1, \dots, p, \quad j, k = 1, \dots, p, \quad \mu, \nu = 0, 1, \dots, D-1.$$

It is an obvious generalization of the flat space-time action given in [65].

To prove the invariance of the action under infinitesimal diffeomorphisms on the world-volume (reparametrizations), we first write down the corresponding transformation law for the (r,s)-type tensor density of weight  $a$

$$\begin{aligned} \delta_\varepsilon T_{K_1 \dots K_s}^{J_1 \dots J_r}[a] &= L_\varepsilon T_{K_1 \dots K_s}^{J_1 \dots J_r}[a] = \varepsilon^L \partial_L T_{K_1 \dots K_s}^{J_1 \dots J_r}[a] \\ &+ T_{K K_2 \dots K_s}^{J_1 \dots J_r}[a] \partial_{K_1} \varepsilon^K + \dots + T_{K_1 \dots K_{s-1} K}^{J_1 \dots J_r}[a] \partial_{K_s} \varepsilon^K \\ &- T_{K_1 \dots K_s}^{J J_2 \dots J_r}[a] \partial_J \varepsilon^{J_1} - \dots - T_{K_1 \dots K_s}^{J_1 \dots J_{r-1} J}[a] \partial_J \varepsilon^{J_r} \\ &+ a T_{K_1 \dots K_s}^{J_1 \dots J_r}[a] \partial_L \varepsilon^L, \end{aligned} \quad (5.121)$$

where  $L_\varepsilon$  is the Lie derivative along the vector field  $\varepsilon$ . Using (5.121), one verifies that if  $x^\mu(\xi)$ ,  $g_{\mu\nu}(\xi)$  are world-volume scalars ( $a = 0$ ) and  $V^J(\xi)$  is a world-volume (1,0)-

type tensor density of weight  $a = 1/2$ , then  $\partial_J x^\nu$  is a (0,1)-type tensor,  $\partial_J x^\mu \partial_K x^\nu g_{\mu\nu}$  is a (0,2)-type tensor and  $L$  is a scalar density of weight  $a = 1$ . Therefore,

$$\delta_\varepsilon S = \int d^{p+1} \xi \partial_J (\varepsilon^J L)$$

and this variation vanishes under suitable boundary conditions.

The equations of motion following from (5.120) are:

$$\begin{aligned} \partial_J (V^J V^K \partial_K x^\lambda) + \Gamma_{\mu\nu}^\lambda V^J V^K \partial_J x^\mu \partial_K x^\nu &= 0, \\ V^J \partial_J x^\mu \partial_K x^\nu g_{\mu\nu}(x) &= 0, \end{aligned}$$

where  $\Gamma_{\mu\nu}^\lambda$  is the connection compatible with the metric  $g_{\mu\nu}(x)$ :

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}).$$

For the transition to Hamiltonian picture it is convenient to rewrite the Lagrangian density (5.120) in the form ( $\partial_\tau = \partial/\partial\tau$ ,  $\partial_j = \partial/\partial\sigma^j$ ):

$$L = \frac{1}{4\mu^0} g_{\mu\nu}(x) (\partial_\tau - \mu^j \partial_j) x^\mu (\partial_\tau - \mu^k \partial_k) x^\nu, \quad (5.122)$$

where

$$V^J = (V^0, V^j) = \left( -\frac{1}{2\sqrt{\mu^0}}, \frac{\mu^j}{2\sqrt{\mu^0}} \right).$$

Now the equation of motion for  $x^\nu$  takes the form:

$$\begin{aligned} \partial_\tau \left[ \frac{1}{2\mu^0} (\partial_\tau - \mu^k \partial_k) x^\lambda \right] - \partial_j \left[ \frac{\mu^j}{2\mu^0} (\partial_\tau - \mu^k \partial_k) x^\lambda \right] \\ + \frac{1}{2\mu^0} \Gamma_{\mu\nu}^\lambda (\partial_\tau - \mu^j \partial_j) x^\mu (\partial_\tau - \mu^k \partial_k) x^\nu &= 0. \end{aligned} \quad (5.123)$$

The equations of motion for the Lagrange multipliers  $\mu^0$  and  $\mu^j$  which follow from (5.122) give the constraints :

$$\begin{aligned} g_{\mu\nu}(x) (\partial_\tau - \mu^j \partial_j) x^\mu (\partial_\tau - \mu^k \partial_k) x^\nu &= 0, \\ g_{\mu\nu}(x) (\partial_\tau - \mu^k \partial_k) x^\mu \partial_j x^\nu &= 0. \end{aligned} \quad (5.124)$$

In terms of  $x^\nu$  and the conjugated momentum  $p_\nu$  they read:

$$T_0 = g^{\mu\nu}(x) p_\mu p_\nu = 0, \quad T_j = p_\nu \partial_j x^\nu = 0. \quad (5.125)$$



The Hamiltonian which corresponds to the Lagrangian density (5.122) is a linear combination of the constraints (5.125) :

$$H_0 = \int d^p \sigma (\mu^0 T_0 + \mu^j T_j).$$

They satisfy the following (equal  $\tau$ ) Poisson bracket algebra

$$\begin{aligned} \{T_0(\underline{\sigma}_1), T_0(\underline{\sigma}_2)\} &= 0, \\ \{T_0(\underline{\sigma}_1), T_j(\underline{\sigma}_2)\} &= [T_0(\underline{\sigma}_1) + T_0(\underline{\sigma}_2)] \partial_j \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\ \{T_j(\underline{\sigma}_1), T_k(\underline{\sigma}_2)\} &= [\delta_j^l T_k(\underline{\sigma}_1) + \delta_k^l T_j(\underline{\sigma}_2)] \partial_l \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\ \underline{\sigma} &= (\sigma^1, \dots, \sigma^p). \end{aligned} \tag{5.126}$$

The equalities (5.126) show that the constraint algebra is the same for flat and for curved backgrounds. Consequently, one can apply the BFV-BRST approach in exactly the same way as in the previous section.

## 5.2 Null membranes in D=4

Here we confine ourselves to the case of membranes moving in a four dimensional, stationary, axially symmetric gravity background of the type

$$\begin{aligned} ds^2 &= g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + 2g_{03}dx^0 dx^3, \\ g_{\mu\nu} &= g_{\mu\nu}(x^1, x^2). \end{aligned} \tag{5.127}$$

We will work in the gauge  $\mu^0, \mu^j = \text{constants}$ , in which the equations of motion (5.123) and constraints (5.124) for the membrane ( $j, k = 1, 2$ ) have the form:

$$(\partial_\tau - \mu^j \partial_j)(\partial_\tau - \mu^k \partial_k)x^\lambda + \Gamma_{\mu\nu}^\lambda (\partial_\tau - \mu^j \partial_j)x^\mu (\partial_\tau - \mu^k \partial_k)x^\nu = 0. \tag{5.128}$$

$$g_{\mu\nu}(x)(\partial_\tau - \mu^j \partial_j)x^\mu (\partial_\tau - \mu^k \partial_k)x^\nu = 0, \tag{5.129}$$

$$g_{\mu\nu}(x)(\partial_\tau - \mu^k \partial_k)x^\mu \partial_j x^\nu = 0.$$

To establish the correspondence with the null geodesics we note that if we introduce the quantities

$$u^\nu(x) = (\partial_\tau - \mu^j \partial_j)x^\nu, \tag{5.130}$$

the equations of motion (5.128) can be rewritten as

$$u^\nu(\partial_\nu u^\lambda + \Gamma_{\mu\nu}^\lambda u^\mu) = 0.$$

Then it follows from here that  $u^2$  does not depend on  $x^\nu$ . In this notations, the constraints are:

$$g_{\mu\nu} u^\mu u^\nu = 0 \quad , \quad g_{\mu\nu} u^\mu \partial_j x^\nu = 0.$$

Taking into account the metric (5.127), one can write the equations of motion (5.128) and the constraint (5.129) in the form:

$$\begin{aligned} & (\partial_\tau - \mu^j \partial_j) u^0 + 2(\Gamma_{01}^0 u^0 + \Gamma_{13}^0 u^3)(\partial_\tau - \mu^j \partial_j) x^1 + \\ & + 2(\Gamma_{02}^0 u^0 + \Gamma_{23}^0 u^3)(\partial_\tau - \mu^j \partial_j) x^2 = 0, \\ & (\partial_\tau - \mu^j \partial_j)(\partial_\tau - \mu^k \partial_k) x^1 + \Gamma_{11}^1 (\partial_\tau - \mu^j \partial_j) x^1 (\partial_\tau - \mu^k \partial_k) x^1 + \\ & + 2\Gamma_{12}^1 (\partial_\tau - \mu^j \partial_j) x^1 (\partial_\tau - \mu^k \partial_k) x^2 + \Gamma_{22}^1 (\partial_\tau - \mu^j \partial_j) x^2 (\partial_\tau - \mu^k \partial_k) x^2 + \\ & + \Gamma_{00}^1 (u^0)^2 + 2\Gamma_{03}^1 u^0 u^3 + \Gamma_{33}^1 (u^3)^2 = 0, \\ & (\partial_\tau - \mu^j \partial_j)(\partial_\tau - \mu^k \partial_k) x^2 + \Gamma_{11}^2 (\partial_\tau - \mu^j \partial_j) x^1 (\partial_\tau - \mu^k \partial_k) x^1 + \quad (5.131) \\ & + 2\Gamma_{12}^2 (\partial_\tau - \mu^j \partial_j) x^1 (\partial_\tau - \mu^k \partial_k) x^2 + \Gamma_{22}^2 (\partial_\tau - \mu^j \partial_j) x^2 (\partial_\tau - \mu^k \partial_k) x^2 + \\ & + \Gamma_{00}^2 (u^0)^2 + 2\Gamma_{03}^2 u^0 u^3 + \Gamma_{33}^2 (u^3)^2 = 0, \\ & (\partial_\tau - \mu^j \partial_j) u^3 + 2(\Gamma_{01}^3 u^0 + \Gamma_{13}^3 u^3)(\partial_\tau - \mu^j \partial_j) x^1 + \\ & + 2(\Gamma_{02}^3 u^0 + \Gamma_{23}^3 u^3)(\partial_\tau - \mu^j \partial_j) x^2 = 0, \\ & g_{11}(\partial_\tau - \mu^j \partial_j) x^1 (\partial_\tau - \mu^k \partial_k) x^1 + g_{22}(\partial_\tau - \mu^j \partial_j) x^2 (\partial_\tau - \mu^k \partial_k) x^2 + \\ & + g_{00} (u^0)^2 + 2g_{03} u^0 u^3 + g_{33} (u^3)^2 = 0, \\ & g_{11}(\partial_\tau - \mu^k \partial_k) x^1 \partial_j x^1 + g_{22}(\partial_\tau - \mu^k \partial_k) x^2 \partial_j x^2 + \\ & + (g_{00} \partial_j x^0 + g_{03} \partial_j x^3) u^0 + (g_{03} \partial_j x^0 + g_{33} \partial_j x^3) u^3 = 0, \end{aligned}$$

where the notation introduced in (5.130) is used. To simplify these equations, we use the ansatz

$$\begin{aligned} x^0(\tau, \underline{\sigma}) &= f^0(z^1, z^2) + t(\tau), \\ x^1(\tau, \underline{\sigma}) &= r(\tau) \quad , \quad x^2(\tau, \underline{\sigma}) = \theta(\tau), \end{aligned} \quad (5.132)$$

$$\begin{aligned}x^3(\tau, \underline{z}) &= f^3(z^1, z^2) + \varphi(\tau), \\z^j &= \mu^j \tau + \sigma^j,\end{aligned}$$

where  $f^0, f^3$  are arbitrary functions of their arguments.

After substituting (5.132) in (5.131), we receive (the dot is used for differentiation with respect to the affine parameter  $\tau$ ):

$$\begin{aligned}\dot{u}^0 + \left[ (g^{00} \partial_1 g_{00} + g^{03} \partial_1 g_{03}) u^0 + (g^{00} \partial_1 g_{03} + g^{03} \partial_1 g_{33}) u^3 \right] \dot{r} \\ + \left[ (g^{00} \partial_2 g_{00} + g^{03} \partial_2 g_{03}) u^0 + (g^{00} \partial_2 g_{03} + g^{03} \partial_2 g_{33}) u^3 \right] \dot{\theta} = 0,\end{aligned}\tag{5.133}$$

$$2g_{11} \ddot{r} + \partial_1 g_{11} \dot{r}^2 + 2\partial_2 g_{11} \dot{r} \dot{\theta} - \partial_1 g_{22} \dot{\theta}^2\tag{5.134}$$

$$-[\partial_1 g_{00} (u^0)^2 + 2\partial_1 g_{03} u^0 u^3 + \partial_1 g_{33} (u^3)^2] = 0,$$

$$2g_{22} \ddot{\theta} + \partial_2 g_{22} \dot{\theta}^2 + 2\partial_1 g_{22} \dot{r} \dot{\theta} - \partial_2 g_{11} \dot{r}^2\tag{5.135}$$

$$-[\partial_2 g_{00} (u^0)^2 + 2\partial_2 g_{03} u^0 u^3 + \partial_2 g_{33} (u^3)^2] = 0,$$

$$\begin{aligned}\dot{u}^3 + \left[ (g^{33} \partial_1 g_{03} + g^{03} \partial_1 g_{00}) u^0 + (g^{33} \partial_1 g_{33} + g^{03} \partial_1 g_{03}) u^3 \right] \dot{r} \\ + \left[ (g^{33} \partial_2 g_{03} + g^{03} \partial_2 g_{00}) u^0 + (g^{33} \partial_2 g_{33} + g^{03} \partial_2 g_{03}) u^3 \right] \dot{\theta} = 0,\end{aligned}\tag{5.136}$$

$$g_{11} \dot{r}^2 + g_{22} \dot{\theta}^2 + g_{00} (u^0)^2 + 2g_{03} u^0 u^3 + g_{33} (u^3)^2 = 0,\tag{5.137}$$

$$(g_{00} \partial_j f^0 + g_{03} \partial_j f^3) u^0 + (g_{03} \partial_j f^0 + g_{33} \partial_j f^3) u^3 = 0.\tag{5.138}$$

If we choose

$$f^0(z^1, z^2) = f^0(w) \quad , \quad f^3(z^1, z^2) = f^3(w),$$

where  $w = w(z^1, z^2)$  is an arbitrary function of  $z^1$  and  $z^2$ , then the system of equations (5.138) reduces to the single equation

$$\left( g_{00} \frac{df^0}{dw} + g_{03} \frac{df^3}{dw} \right) u^0 + \left( g_{03} \frac{df^0}{dw} + g_{33} \frac{df^3}{dw} \right) u^3 = 0.\tag{5.139}$$

To be able to separate the variables  $u^0, u^3$  in the system of differential equations (5.133), (5.136) with the help of (5.139), we impose the following condition on  $f^0(w)$  and  $f^3(w)$

$$f^0(w) = C^0 f[w(z^1, z^2)] \quad , \quad f^3(w) = C^3 f[w(z^1, z^2)],$$

where  $C^0, C^3$  are constants, and  $f(w)$  is an arbitrary function of  $w$ . Then the solution of (5.133), (5.136) and (5.139) is [103] ( $C_1 = \text{const}$ ):

$$\begin{aligned} u^0(\tau) &= -C_1 (C^0 g_{03} + C^3 g_{33}) \exp(-H), \\ u^3(\tau) &= +C_1 (C^0 g_{00} + C^3 g_{03}) \exp(-H), \\ H &= \int (g^{00} dg_{00} + 2g^{03} dg_{03} + g^{33} dg_{33}). \end{aligned} \quad (5.140)$$

The condition for the compatibility of (5.140) with (5.134), (5.135) and (5.137) is:

$$\begin{aligned} u^0(\tau) &= -C_1 (C^0 g_{03} + C^3 g_{33}) h^{-1} \\ &= -C_1 (C^3 g^{00} - C^0 g^{03}) = \dot{t}(\tau), \\ u^3(\tau) &= +C_1 (C^0 g_{00} + C^3 g_{03}) h^{-1} \\ &= -C_1 (C^3 g^{03} - C^0 g^{33}) = \dot{\varphi}(\tau), \\ h &= g_{00} g_{33} - g_{03}^2. \end{aligned} \quad (5.141)$$

On the other hand, from (5.135) and (5.137) one has:

$$\begin{aligned} \dot{r}^2 &= -g^{11} \left[ C_1^2 \frac{G}{h} + g^{22} (g_{22}^2 \dot{\theta}^2) \right] \\ &= g^{11} \left\{ C_1^2 [2C^0 C^3 g^{03} - (C^3)^2 g^{00} - (C^0)^2 g^{33}] - g^{22} (g_{22}^2 \dot{\theta}^2) \right\}, \end{aligned} \quad (5.142)$$

$$\begin{aligned} g_{22}^2 \dot{\theta}^2 &= C_2 + C_1^2 \int^\theta d\theta h^{-2} \left[ g_{22} G \frac{\partial h}{\partial \theta} - h \frac{\partial g_{22} G}{\partial \theta} \right], \\ G &= (C^0)^2 g_{00} + 2C^0 C^3 g_{03} + (C^3)^2 g_{33}. \end{aligned} \quad (5.143)$$

In obtaining (5.143), we have used the gauge freedom in the metric (5.127), to impose the condition [104]:

$$\partial_2 \left( \frac{g_{22}}{g_{11}} \right) = 0.$$

As a final result we have

$$\begin{aligned} x^0 &= C^0 f[w(z^1, z^2)] + t(\tau), \\ x^1 &= r(\tau), \\ x^2 &= \theta(\tau), \\ x^3 &= C^3 f[w(z^1, z^2)] + \varphi(\tau), \end{aligned}$$

where  $\dot{t}(\tau), \dot{r}(\tau), \dot{\theta}(\tau), \dot{\varphi}(\tau)$  are given by (5.141), (5.142), and (5.143).

In the particular case when  $x^2 = \theta = \theta_0 = \text{const}$ , one can integrate to obtain the following exact solution of the equations of motion and constraints for the null membrane in the gravity background (5.127):

$$\begin{aligned}
x^0(\tau, \sigma^1, \sigma^2) &= C^0 f[w(z^1, z^2)] + t_0 \\
&\pm \int_{r_0}^r dr \left( C^3 g^{00} - C^0 g^{03} \right) W^{-1/2}, \\
x^3(\tau, \sigma^1, \sigma^2) &= C^3 f[w(z^1, z^2)] + \varphi_0 \\
&\pm \int_{r_0}^r dr \left( C^3 g^{03} - C^0 g^{33} \right) W^{-1/2}, \\
C_1(\tau - \tau_0) &= \pm \int_{r_0}^r dr W^{-1/2}, \\
W &= g^{11} \left[ 2C^0 C^3 g^{03} - (C^3)^2 g^{00} - (C^0)^2 g^{33} \right], \\
t_0, r_0, \varphi_0, \tau_0 &= \text{constants}.
\end{aligned} \tag{5.144}$$

### 5.3 Examples

Here we give some examples of solutions of the type received in the previous section. To begin with, let us start with the simplest case of *Minkowski space-time*. The metric is

$$g_{00} = -1, \quad g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2 \theta,$$

and equalities (5.141), (5.142), (5.143) take the form:

$$\begin{aligned}
\dot{t} &= C_1 C^3, \\
\dot{r}^2 &= (C_1 C^3)^2 - \frac{C_2}{r^2}, \\
r^4 \dot{\theta}^2 &= C_2 - \frac{(C_1 C^0)^2}{\sin^2 \theta}, \\
\dot{\varphi} &= \frac{C_1 C^0}{r^2 \sin^2 \theta}.
\end{aligned}$$

When  $\theta = \theta_0 = \text{const}$ , the solution (5.144) is:

$$x^0(\tau, \sigma^1, \sigma^2) = C^0 f[w(z^1, z^2)] + t_0 \mp C^3 \int_{r_0}^r \frac{dr}{[(C^3)^2 - (C^0)^2 r^{-2} \sin^{-2} \theta_0]^{1/2}},$$

$$\begin{aligned}
x^3(\tau, \sigma^1, \sigma^2) &= C^3 f[w(z^1, z^2)] + \varphi_0 \mp \frac{C^0}{\sin^2 \theta_0} \int_{r_0}^r \frac{dr}{r^2 [(C^3)^2 - (C^0)^2 r^{-2} \sin^{-2} \theta_0]^{1/2}}, \\
C_1(\tau - \tau_0) &= \pm \int_{r_0}^r \frac{dr}{[(C^3)^2 - (C^0)^2 r^{-2} \sin^{-2} \theta_0]^{1/2}}.
\end{aligned}$$

Our next example is the *de Sitter space-time*. We take the metric in the form

$$g_{00} = -(1 - kr^2), g_{11} = (1 - kr^2)^{-1}, g_{22} = r^2, g_{33} = r^2 \sin^2 \theta,$$

where  $k$  is the constant curvature. Now we have

$$\begin{aligned}
\dot{t} &= \frac{C_1 C^3}{1 - kr^2}, \\
\dot{r}^2 &= (C_1 C^3)^2 + C_2 (k - r^{-2}), \\
r^4 \dot{\theta}^2 &= C_2 - \frac{(C_1 C^0)^2}{\sin^2 \theta}, \\
\dot{\varphi} &= \frac{C_1 C^0}{r^2 \sin^2 \theta},
\end{aligned}$$

and the corresponding solution (5.144) is:

$$\begin{aligned}
x^0(\tau, \sigma^1, \sigma^2) &= C^0 f[w(z^1, z^2)] + t_0 \\
&\mp C^3 \int_{r_0}^r \frac{dr}{(1 - kr^2) [(C^3)^2 + (C^0)^2 (k - r^{-2}) \sin^{-2} \theta_0]^{1/2}}, \\
x^3(\tau, \sigma^1, \sigma^2) &= C^3 f[w(z^1, z^2)] + \varphi_0 \\
&\mp \frac{C^0}{\sin^2 \theta_0} \int_{r_0}^r \frac{dr}{r^2 [(C^3)^2 + (C^0)^2 (k - r^{-2}) \sin^{-2} \theta_0]^{1/2}}, \\
C_1(\tau - \tau_0) &= \pm \int_{r_0}^r \frac{dr}{[(C^3)^2 + (C^0)^2 (k - r^{-2}) \sin^{-2} \theta_0]^{1/2}}.
\end{aligned}$$

Now let us turn to the case of *Schwarzschild space-time*. The corresponding metric may be written as

$$\begin{aligned}
g_{00} &= -(1 - 2Mr^{-1}), & g_{11} &= (1 - 2Mr^{-1})^{-1}, \\
g_{22} &= r^2, & g_{33} &= r^2 \sin^2 \theta,
\end{aligned}$$

where  $M$  is the Schwarzschild mass. The equalities (5.141), (5.142) and (5.143) read

$$\dot{t} = \frac{C_1 C^3}{1 - 2Mr^{-1}},$$

$$\begin{aligned}
\dot{r}^2 &= (C_1 C^3)^2 - \frac{C_2}{r^2} (1 - 2Mr^{-1}), \\
r^4 \dot{\theta}^2 &= C_2 - \frac{(C_1 C^0)^2}{\sin^2 \theta}, \\
\dot{\varphi} &= \frac{C_1 C^0}{r^2 \sin^2 \theta}.
\end{aligned} \tag{5.145}$$

When  $\theta = \theta_0 = \text{const}$ , one obtains from (5.144)

$$\begin{aligned}
x^0(\tau, \sigma^1, \sigma^2) &= C^0 f[w(z^1, z^2)] + t_0 \\
&\mp C^3 \int_{r_0}^r \frac{dr}{(1 - 2Mr^{-1})[(C^3)^2 - (C^0)^2 r^{-2} (1 - 2Mr^{-1}) \sin^{-2} \theta_0]^{1/2}}, \\
x^3(\tau, \sigma^1, \sigma^2) &= C^3 f[w(z^1, z^2)] + \varphi_0 \\
&\mp \frac{C^0}{\sin^2 \theta_0} \int_{r_0}^r \frac{dr}{r^2 [(C^3)^2 - (C^0)^2 r^{-2} (1 - 2Mr^{-1}) \sin^{-2} \theta_0]^{1/2}}, \\
C_1(\tau - \tau_0) &= \pm \int_{r_0}^r \frac{dr}{[(C^3)^2 - (C^0)^2 r^{-2} (1 - 2Mr^{-1}) \sin^{-2} \theta_0]^{1/2}}.
\end{aligned}$$

For the *Taub-NUT space-time* we take the metric as

$$\begin{aligned}
g_{00} &= -\frac{\delta}{R^2}, & g_{11} &= \frac{R^2}{\delta}, & g_{22} &= R^2, \\
g_{33} &= R^2 \sin^2 \theta - 4l^2 \frac{\delta \cos^2 \theta}{R^2}, & g_{03} &= -2l \frac{\delta \cos \theta}{R^2}, \\
\delta(r) &= r^2 - 2Mr - l^2, & R^2(r) &= r^2 + l^2,
\end{aligned}$$

where  $M$  is the mass and  $l$  is the NUT-parameter. Now we have from (5.141), (5.142) and (5.143):

$$\begin{aligned}
\dot{t} &= \frac{C_1}{R^2} \left[ C^3 \left( \frac{R^4}{\delta} + 4l^2 \right) - \frac{2l}{\sin^2 \theta} (C^0 \cos \theta + 2C^3 l) \right], \\
R^4 \dot{r}^2 &= (C_1 C^3)^2 (R^4 + 4l^2 \delta) - C_2 \delta, \\
R^4 \dot{\theta}^2 &= C_2 - \frac{C_1^2}{\sin^2 \theta} [(C^0)^2 + (2C^3 l)^2 + 4C^0 C^3 l \cos \theta], \\
\dot{\varphi} &= \frac{C_1}{R^2 \sin^2 \theta} (C^0 + 2C^3 l \cos \theta).
\end{aligned}$$

In the Taub-NUT metric the solution (5.144) is

$$\begin{aligned}
x^0(\tau, \sigma^1, \sigma^2) &= C^0 f[w(z^1, z^2)] + t_0 \\
&\pm \int_{r_0}^r dr \left[ C^3 (R^4 \delta^{-1} + 4l^2) - 2l \sin^{-2} \theta_0 (C^0 \cos \theta_0 + 2C^3 l) \right] U^{-1/2}(r),
\end{aligned}$$

$$\begin{aligned}
x^3(\tau, \sigma^1, \sigma^2) &= C^3 f[w(z^1, z^2)] + \varphi_0 \\
&\pm \frac{1}{\sin^2 \theta_0} \int_{r_0}^r dr \left( C^0 + 2C^3 l \cos \theta_0 \right) U^{-1/2}(r), \\
C_1(\tau - \tau_0) &= \pm \int_{r_0}^r dr R^2 U^{-1/2}(r),
\end{aligned}$$

where

$$U(r) = \left( C^3 \right)^2 \left( R^4 - 4l^2 \delta \cot^2 \theta_0 \right) - \delta \sin^{-2} \theta_0 \left[ \left( C^0 \right)^2 + 4C^0 C^3 l \cos \theta_0 \right].$$

Finally, we consider the *Kerr space-time* with metric taken in the form

$$\begin{aligned}
g_{00} &= - \left( 1 - \frac{2Mr}{\rho^2} \right) \quad , \quad g_{11} = \frac{\rho^2}{\Delta}, \\
g_{22} &= \rho^2 \quad , \quad g_{33} = \left( r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\rho^2} \right) \sin^2 \theta, \\
g_{03} &= - \frac{2Mar \sin^2 \theta}{\rho^2},
\end{aligned}$$

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta \quad , \quad \Delta = r^2 - 2Mr + a^2,$$

$M$  is the mass and  $a$  is the angular momentum per unit mass of the Kerr black hole.

With this input in equations (5.141), (5.142) and (5.143), we have:

$$\begin{aligned}
\dot{t} &= \frac{C_1}{\Delta \rho^2} \left[ C^3 (r^2 + a^2)^2 - C^3 a^2 \Delta \sin^2 \theta - 2C^0 Mar \right], \\
\rho^4 \dot{r}^2 &= C_1^2 \left[ (C^3)^2 (r^2 + a^2)^2 - 4C^0 C^3 Mar + (C^0)^2 a^2 \right] - C_2 \Delta, \\
\rho^4 \dot{\theta}^2 &= C_2 - C_1^2 \left( \frac{(C^0)^2}{\sin^2 \theta} + (C^3)^2 a^2 \sin^2 \theta \right) \\
\dot{\varphi} &= \frac{C_1}{\Delta \rho^2} \left( \frac{C^0 \Delta}{\sin^2 \theta} + 2C^3 Mar - C^0 a^2 \right).
\end{aligned}$$

In this case, the exact solution (5.144) takes the form:

$$\begin{aligned}
x^0(\tau, \sigma^1, \sigma^2) &= C^0 f[w(z^1, z^2)] + t_0 \\
&\pm \int_{r_0}^r dr \Delta^{-1} \left\{ 2C^0 Mar - C^3 \left[ (r^2 + a^2) \rho_0^2 + 2Ma^2 r \sin^2 \theta_0 \right] \right\} V^{-1/2}(r), \\
x^3(\tau, \sigma^1, \sigma^2) &= C^3 f[w(z^1, z^2)] + \varphi_0
\end{aligned} \tag{5.146}$$



$$\begin{aligned}
& \pm \frac{1}{\sin^2 \theta_0} \int_{r_0}^r dr \Delta^{-1} \left[ C^0 (2Mr - \rho_0^2) - 2C^3 Mar \sin^2 \theta_0 \right] V^{-1/2}(r), \\
C_1(\tau - \tau_0) &= \pm \int_{r_0}^r dr \rho_0^2 V^{-1/2}(r), \\
V(r) &= (C^0)^2 (a^2 - \Delta \sin^2 \theta_0) - 4C^0 C^3 Mar \\
&+ (C^3)^2 [(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta_0], \\
\rho_0^2 &= r^2 + a^2 \cos^2 \theta_0.
\end{aligned}$$

## 5.4 Comments

In the previous subsection we restrict ourselves to some particular cases of the generic solution (5.144) and did not pay attention to the existing possibilities for obtaining solutions in the case  $\theta \neq \text{const}$  in the considered examples.

Obviously, the examples given in the previous subsection do not exhaust all possibilities contained in the metric (5.127) [105]. On the other hand, in different particular cases of this type of metric, there exist more general brane solutions. They will be not considered here. We only mention that in the gauge  $\mu^k = \text{const}$ ,

$$x^\nu(\tau, \underline{\sigma}) = x^\nu(\mu^k \tau + \sigma^k)$$

is an obvious nontrivial solution of the equations of motion and of the constraints (5.123), (5.124) depending on  $D$  arbitrary functions of  $p$  variables for the null  $p$ -brane in arbitrary  $D$ -dimensional gravity background.

From the results of the previous subsection, it is easy to extract the corresponding formulas for the null string case simply by putting  $\sigma^1 = \sigma, \mu^1 = \mu, \sigma^2 = \mu^2 = 0$ . For example, our equalities (5.145) coincide with the ones obtained in [106] for the null string moving in Schwarzschild space-time after identification:

$$E = C_1 C^3, \quad L = C_1 C^0, \quad L^2 + K = C_2.$$

Moreover, our solution (5.146) in the case  $p = 1$ , generalizes the solution given in [103]. The latter corresponds to fixing the arbitrary function  $f(w)$  to a linear one

and fixing the gauge to  $\mu = 0$ , i.e.

$$f[w(\mu^1\tau + \sigma^1, \mu^2\tau + \sigma^2)] \mapsto f(\mu\tau + \sigma) = \mu\tau + \sigma, \quad \text{with} \quad \mu = 0.$$

## 6 D=10 CHIRAL NULL SUPER p-BRANES

Here we consider a model for tensionless super  $p$ -branes with  $N$  chiral supersymmetries in ten dimensional flat space-time. After establishing the symmetries of the action, we give the general solution of the classical equations of motion in a particular gauge. In the case of a null superstring ( $p=1$ ) we find the general solution in an arbitrary gauge. Then, using a harmonic superspace approach, the initial algebra of first and second class constraints is converted into an algebra of Lorentz-covariant, BFV-irreducible, first class constraints only. The corresponding BRST charge is as for a first rank dynamical system. This section is based on the papers [67, 68, 69].

### 6.1 Lagrangian formulation

We define our model for  $D = 10$   $N$ -extended chiral tensionless super  $p$ -branes by the action:

$$\begin{aligned}
 S &= \int d^{p+1} \xi L \quad , \quad L = V^J V^K \Pi_J^\mu \Pi_K^\nu \eta_{\mu\nu}, \quad (6.147) \\
 \Pi_J^\mu &= \partial_J x^\mu + i \sum_{A=1}^N (\theta^A \sigma^\mu \partial_J \theta^A) \quad , \quad \partial_J = \partial / \partial \xi^J, \\
 \xi^J &= (\xi^0, \xi^j) = (\tau, \sigma^j), \quad \text{diag}(\eta_{\mu\nu}) = (-, +, \dots, +), \\
 J, K &= 0, 1, \dots, p \quad , \quad j, k = 1, \dots, p \quad , \quad \mu, \nu = 0, 1, \dots, 9.
 \end{aligned}$$

Here  $(x^\nu, \theta^{A\alpha})$  are the superspace coordinates,  $\theta^{A\alpha}$  are  $N$  left Majorana-Weyl space-time spinors ( $\alpha = 1, \dots, 16$ ,  $N$  being the number of the supersymmetries) and  $\sigma^\mu$  are the 10-dimensional Pauli matrices (our spinor conventions are given in the Appendix B). Actions of this type are first given in [107] for the case of tensionless superstring ( $p = 1, N = 1$ ) and in [65] for the bosonic case ( $N = 0$ ).

The action (6.147) has an obvious global Poincaré invariance. Under global infinitesimal supersymmetry transformations, the fields  $\theta^{A\alpha}(\xi)$ ,  $x^\nu(\xi)$  and  $V^J(\xi)$

transform as follows:

$$\delta_\eta^{A\alpha} = \eta^{A\alpha} \quad , \quad \delta_\eta x^\mu = i \sum_A (\theta^A \sigma^\mu \delta_\eta \theta^A) \quad , \quad \delta_\eta V^J = 0.$$

As a consequence  $\delta_\eta \Pi_J^\mu = 0$  and hence also  $\delta_\eta L = \delta_\eta S = 0$ .

Using (5.121), one verifies that if  $x^\mu(\xi)$ ,  $\theta^{A\alpha}(\xi)$  are world-volume scalars ( $a = 0$ ) and  $V^J(\xi)$  is a world-volume (1,0)-type tensor density of weight  $a = 1/2$ , then  $\Pi_J^\nu$  is a (0,1)-type tensor,  $\Pi_J^\nu \Pi_{K\nu}$  is a (0,2)-type tensor and  $L$  is a scalar density of weight  $a = 1$ . Therefore,

$$\delta_\varepsilon S = \int d^{p+1} \xi \partial_J (\varepsilon^J L)$$

and this variation vanishes under suitable boundary conditions.

Let us now check the  $\kappa$ -invariance of the action. We define the  $\kappa$ -variations of  $\theta^{A\alpha}(\xi)$ ,  $x^\nu(\xi)$  and  $V^J(\xi)$  as follows:

$$\begin{aligned} \delta_\kappa \theta^{A\alpha} &= i(\Gamma \kappa^A)^\alpha = iV^J (\Pi_J \kappa^A)^\alpha, & \delta_\kappa x^\nu &= -i \sum_A (\theta^A \sigma^\nu \delta_\kappa \theta^A), \\ \delta_\kappa V^K &= 2V^K V^L \sum_A (\partial_L \theta^A \kappa^A). \end{aligned} \quad (6.148)$$

Therefore,  $\kappa^{A\alpha}(\xi)$  are left Majorana-Weyl space-time spinors and world-volume scalar densities of weight  $a = -1/2$ .

From (6.148) we obtain:

$$\delta_\kappa (\Pi_J^\nu \Pi_{K\nu}) = -2i \sum_A [\partial_J \theta^A \Pi_K + \partial_K \theta^A \Pi_J] \delta_\kappa \theta^A$$

and

$$\delta_\kappa L = 2V^J \Pi_J^\nu \Pi_{K\nu} [\delta_\kappa V^K - 2V^K V^L \sum_A (\partial_L \theta^A \kappa^A)] = 0.$$

The algebra of kappa-transformations closes only on the equations of motion, which can be written in the form:

$$\partial_J (V^J V^K \Pi_{K\nu}) = 0, \quad V^J V^K (\partial_J \theta^A \Pi_K)_\alpha = 0, \quad V^J \Pi_J^\nu \Pi_{K\nu} = 0. \quad (6.149)$$

As usual, an additional local bosonic world-volume symmetry is needed for its closure. In our case, the Lagrangian, and therefore the action, are invariant under the

following transformations of the fields:

$$\delta_\lambda \theta^A(\xi) = \lambda V^J \partial_J \theta^A, \quad \delta_\lambda x^\nu(\xi) = -i \sum_A (\theta^A \sigma^\nu \delta_\lambda \theta^A), \quad \delta_\lambda V^J(\xi) = 0.$$

Now, checking the commutator of two  $\kappa$ -transformations, we find:

$$\begin{aligned} [\delta_{\kappa_1}, \delta_{\kappa_2}] \theta^{A\alpha}(\xi) &= \delta_\kappa \theta^{A\alpha}(\xi) + \text{terms} \propto \text{eqs. of motion}, \\ [\delta_{\kappa_1}, \delta_{\kappa_2}] x^\nu(\xi) &= (\delta_\kappa + \delta_\varepsilon + \delta_\lambda) x^\nu(\xi) + \text{terms} \propto \text{eqs. of motion}, \\ [\delta_{\kappa_1}, \delta_{\kappa_2}] V^J(\xi) &= \delta_\varepsilon V^J(\xi) + \text{terms} \propto \text{eqs. of motion}. \end{aligned}$$

Here  $\kappa^{A\alpha}(\xi)$ ,  $\lambda(\xi)$  and  $\varepsilon(\xi)$  are given by the expressions:

$$\begin{aligned} \kappa^{A\alpha} &= -2V^K \sum_B [(\partial_K \theta^B \kappa_1^B) \kappa_2^{A\alpha} - (\partial_K \theta^B \kappa_2^B) \kappa_1^{A\alpha}], \\ \lambda &= 4iV^K \sum_A (\kappa_1^A \Pi_K \kappa_2^A), \quad \varepsilon^J = -V^J \lambda. \end{aligned}$$

We note that  $\Gamma_{\alpha\beta} = (V^J \Pi_J)_{\alpha\beta}$  in (6.148) has the following property on the equations of motion

$$\Gamma^2 = 0.$$

This means, that the local  $\kappa$ -invariance of the action indeed eliminates half of the components of  $\theta^A$  as is needed.

For transition to Hamiltonian picture, it is convenient to rewrite the Lagrangian density (6.147) in the form  $(\partial_\tau = \partial/\partial\tau, \partial_j = \partial/\partial\sigma^j)$ :

$$L = \frac{1}{4\mu^0} \left[ (\partial_\tau - \mu^j \partial_j) x + i \sum_A \theta^A \sigma (\partial_\tau - \mu^j \partial_j) \theta^A \right]^2, \quad (6.150)$$

where

$$V^J = (V^0, V^j) = \left( -\frac{1}{2\sqrt{\mu^0}}, \frac{\mu^j}{2\sqrt{\mu^0}} \right).$$

The equations of motion for the Lagrange multipliers  $\mu^0$  and  $\mu^j$  which follow from (6.150) give the constraints ( $p_\nu$  and  $p_{\theta\alpha}^A$  are the momenta conjugated to  $x^\nu$  and  $\theta^{A\alpha}$ ):

$$T_0 = p^2, \quad T_j = p_\nu \partial_j x^\nu + \sum_A p_{\theta\alpha}^A \partial_j \theta^{A\alpha}. \quad (6.151)$$

The remaining constraints follow from the definition of the momenta  $p_{\theta\alpha}^A$ :

$$D_\alpha^A = -ip_{\theta\alpha}^A - (\not{x}\theta^A)_\alpha. \quad (6.152)$$

## 6.2 Hamiltonian formulation

The Hamiltonian which corresponds to the Lagrangian density (6.150) is a linear combination of the constraint (6.151) and (6.152):

$$H_0 = \int d^p \sigma [\mu^0 T_0 + \mu^j T_j + \sum_A \mu^{A\alpha} D_\alpha^A] \quad (6.153)$$

It is a generalization of the Hamiltonians for the bosonic null  $p$ -brane and for the  $N$ -extended Green-Schwarz superparticle.

The equations of motion which follow from the Hamiltonian (6.153) are:

$$\begin{aligned} (\partial_\tau - \mu^j \partial_j) x^\nu &= 2\mu^0 p^\nu - \sum_A (\mu^A \sigma^\nu \theta^A), & (\partial_\tau - \mu^j \partial_j) p_\nu &= (\partial_j \mu^j) p_\nu, \\ (\partial_\tau - \mu^j \partial_j) \theta^{A\alpha} &= i\mu^{A\alpha}, & (\partial_\tau - \mu^j \partial_j) p_{\theta\alpha}^A &= (\partial_j \mu^j) p_{\theta\alpha}^A + (\mu^A \not{\partial})_\alpha. \end{aligned} \quad (6.154)$$

In (6.154), one can consider  $\mu^0$ ,  $\mu^j$  and  $\mu^{A\alpha}$  as depending only on  $\underline{\sigma} = (\sigma^1, \dots, \sigma^p)$ , but not on  $\tau$  as a consequence from their equations of motion.

In the gauge when  $\mu^0$ ,  $\mu^j$  and  $\mu^{A\alpha}$  are constants, the general solution of (6.154) is

$$\begin{aligned} x^\nu(\tau, \underline{\sigma}) &= x^\nu(\underline{z}) + \tau [2\mu^0 p^\nu(\underline{z}) - \sum_A (\mu^A \sigma^\nu \theta^A(\tau, \underline{\sigma}))], \\ &= x^\nu(\underline{z}) + \tau [2\mu^0 p^\nu(\underline{z}) - \sum_A (\mu^A \sigma^\nu \theta^A(\underline{z}))] \\ p_\nu(\tau, \underline{\sigma}) &= p_\nu(\underline{z}), & \theta^{A\alpha}(\tau, \underline{\sigma}) &= \theta^{A\alpha}(\underline{z}) + i\tau \mu^{A\alpha}, \\ p_{\theta\alpha}^A(\tau, \underline{\sigma}) &= p_{\theta\alpha}^A(\underline{z}) + \tau (\mu^A \sigma^\nu)_\alpha p_\nu(\underline{z}), \end{aligned} \quad (6.155)$$

where  $x^\nu(\underline{z})$ ,  $p_\nu(\underline{z})$ ,  $\theta^{A\alpha}(\underline{z})$  and  $p_{\theta\alpha}^A(\underline{z})$  are arbitrary functions of their arguments

$$z^j = \mu^j \tau + \sigma^j.$$

In the case of null strings ( $p = 1$ ), one can write explicitly the general solution of the equations of motion in an arbitrary gauge:  $\mu^0 = \mu^0(\sigma)$ ,  $\mu^1 \equiv \mu = \mu(\sigma)$ ,  $\mu^{A\alpha} = \mu^{A\alpha}(\sigma)$ . This solution is given by

$$\begin{aligned} x^\nu(\tau, \sigma) &= g^\nu(w) - 2 \int^\sigma \frac{\mu^0(s)}{\mu^2(s)} ds f^\nu(w) + \sum_A \int^\sigma \frac{\mu^{A\alpha}(s)}{\mu(s)} ds [\sigma^\nu \zeta^A(w)]_\alpha \\ &\quad - i \sum_A \int^\sigma ds_1 \frac{(\mu^A \sigma^\nu)_\alpha(s_1)}{\mu(s_1)} \int^{s_1} \frac{\mu^{A\alpha}(s)}{\mu(s)} ds, \end{aligned}$$

$$\begin{aligned}
p_\nu(\tau, \sigma) &= \mu^{-1}(\sigma) f_\nu(w), \\
\theta^{A\alpha}(\tau, \sigma) &= \zeta^{A\alpha}(w) - i \int_{\mu(s)}^{\sigma} \frac{\mu^{A\alpha}(s)}{\mu(s)} ds, \\
p_{\theta_\alpha}^A(\tau, \sigma) &= \mu^{-1}(\sigma) \left[ h_\alpha^A(w) - \int_{\mu(s)}^{\sigma} \frac{(\mu^A \sigma^\nu)_\alpha(s)}{\mu(s)} ds f_\nu(w) \right].
\end{aligned} \tag{6.156}$$

Here  $g^\nu(w)$ ,  $f_\nu(w)$ ,  $\zeta^{A\alpha}(w)$  and  $h_\alpha^A(w)$  are arbitrary functions of the variable

$$w = \tau + \int \frac{ds}{\mu(s)}$$

The solution (6.155) at  $p = 1$  differs from (6.156) by the choice of the particular solutions of the inhomogeneous equations. As for  $z$  and  $w$ , one can write for example ( $\mu^0$ ,  $\mu$ ,  $\mu^{A\alpha}$  are now constants)

$$p_\nu(\tau, \sigma) = \mu^{-1} f_\nu(\tau + \sigma/\mu) = \mu^{-1} f_\nu[\mu^{-1}(\mu\tau + \sigma)] = p_\nu(z)$$

and analogously for the other arbitrary functions in the general solution of the equations of motion.

Let us now consider the properties of the constraints (6.151), (6.152). They satisfy the following (equal  $\tau$ ) Poisson bracket algebra

$$\begin{aligned}
\{T_0(\underline{\sigma}_1), T_0(\underline{\sigma}_2)\} &= 0, \quad \{T_0(\underline{\sigma}_1), D_\alpha^A(\underline{\sigma}_2)\} = 0, \\
\{T_0(\underline{\sigma}_1), T_j(\underline{\sigma}_2)\} &= [T_0(\underline{\sigma}_1) + T_0(\underline{\sigma}_2)] \partial_j \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\
\{T_j(\underline{\sigma}_1), T_k(\underline{\sigma}_2)\} &= [\delta_j^l T_k(\underline{\sigma}_1) + \delta_k^l T_j(\underline{\sigma}_2)] \partial_l \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\
\{T_j(\underline{\sigma}_1), D_\alpha^A(\underline{\sigma}_2)\} &= D_\alpha^A(\underline{\sigma}_1) \partial_j \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\
\{D_\alpha^A(\underline{\sigma}_1), D_\beta^B(\underline{\sigma}_2)\} &= 2i \delta^{AB} \not p_{\alpha\beta} \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2).
\end{aligned}$$

From the condition that the constraints must be maintained in time, i.e. [108]

$$\{T_0, H_0\} \approx 0, \quad \{T_j, H_0\} \approx 0, \quad \{D_\alpha^A, H_0\} \approx 0, \tag{6.157}$$

it follows that in the Hamiltonian  $H_0$  one has to include the constraints

$$T_\alpha^A = \not p_{\alpha\beta} D^{A\beta}$$

instead of  $D_\alpha^A$ . This is because the Hamiltonian has to be first class quantity, but  $D_\alpha^A$  are a mixture of first and second class constraints.  $T_\alpha^A$  has the following non-zero

Poisson brackets

$$\begin{aligned}\{T_j(\underline{\sigma}_1), T_\alpha^A(\underline{\sigma}_2)\} &= [T_\alpha^A(\underline{\sigma}_1) + T_\alpha^A(\underline{\sigma}_2)]\partial_j\delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\ \{T_\alpha^A(\underline{\sigma}_1), T_\beta^B(\underline{\sigma}_2)\} &= 2i\delta^{AB}\not{p}_{\alpha\beta}T_0\delta^p(\underline{\sigma}_1 - \underline{\sigma}_2).\end{aligned}$$

In this form, our constraints are first class and the Dirac consistency conditions (6.157) (with  $D_\alpha^A$  replaced by  $T_\alpha^A$ ) are satisfied identically. However, one now encounters a new problem. The constraints  $T_0$ ,  $T_j$  and  $T_\alpha^A$  are not BFV-irreducible, i.e. they are functionally dependent:

$$(\not{p}T^A)^\alpha - D^{A\alpha}T_0 = 0.$$

It is known, that in this case after BRST-BFV quantization an infinite number of ghosts for ghosts appear, if one wants to preserve the manifest Lorentz invariance. The way out consists in the introduction of auxiliary variables, so that the mixture of first and second class constraints  $D^{A\alpha}$  can be appropriately covariantly decomposed into first class constraints and second class ones. To this end, here we will use the auxiliary harmonic variables introduced in [109] and [110]. These are  $u_\mu^a$  and  $v_\alpha^\pm$ , where superscripts  $a = 1, \dots, 8$  and  $\pm$  transform under the 'internal' groups  $SO(8)$  and  $SO(1, 1)$  respectively. The just introduced variables are constrained by the following orthogonality conditions

$$u_\mu^a u^{b\mu} = C^{ab}, \quad u_\mu^\pm u^{a\mu} = 0, \quad u_\mu^+ u^{-\mu} = -1,$$

where

$$u_\mu^\pm = v_\alpha^\pm \sigma_\mu^{\alpha\beta} v_\beta^\pm,$$

$C^{ab}$  is the invariant metric tensor in the relevant representation space of  $SO(8)$  and  $(u^\pm)^2 = 0$  as a consequence of the Fierz identity for the 10-dimensional  $\sigma$ -matrices. We note that  $u_\mu^a$  and  $v_\alpha^\pm$  do not depend on  $\underline{\sigma}$ .

Now we have to ensure that our dynamical system does not depend on arbitrary rotations of the auxiliary variables  $(u_\mu^a, u_\mu^\pm)$ . It can be done by introduction of first



class constraints, which generate these transformations

$$\begin{aligned}
I^{ab} &= -(u_\nu^a p_u^{b\nu} - u_\nu^b p_u^{a\nu} + \frac{1}{2} v^+ \sigma^{ab} p_v^+ + \frac{1}{2} v^- \sigma^{ab} p_v^-), & \sigma^{ab} &= u_\mu^a u_\nu^b \sigma^{\mu\nu}, \\
I^{-+} &= -\frac{1}{2} (v_\alpha^+ p_v^{+\alpha} - v_\alpha^- p_v^{-\alpha}), \\
I^{\pm a} &= -(u_\mu^\pm p_u^{a\mu} + \frac{1}{2} v^\mp \sigma^\pm \sigma^a p_v^\mp), & \sigma^\pm &= u_\nu^\pm \sigma^\nu, & \sigma^a &= u_\nu^a \sigma^\nu.
\end{aligned} \tag{6.158}$$

In the above equalities,  $p_u^{a\nu}$  and  $p_v^{\pm\alpha}$  are the momenta canonically conjugated to  $u_\nu^a$  and  $v_\alpha^\pm$ .

The newly introduced constraints (6.158) obey the following Poisson bracket algebra

$$\begin{aligned}
\{I^{ab}, I^{cd}\} &= C^{bc} I^{ad} - C^{ac} I^{bd} + C^{ad} I^{bc} - C^{bd} I^{ac}, \\
\{I^{-+}, I^{\pm a}\} &= \pm I^{\pm a}, \\
\{I^{ab}, I^{\pm c}\} &= C^{bc} I^{\pm a} - C^{ac} I^{\pm b}, \\
\{I^{+a}, I^{-b}\} &= C^{ab} I^{-+} + I^{ab}.
\end{aligned}$$

This algebra is isomorphic to the  $SO(1, 9)$  algebra:  $I^{ab}$  generate  $SO(8)$  rotations,  $I^{-+}$  is the generator of the subgroup  $SO(1, 1)$  and  $I^{\pm a}$  generate the transformations from the coset  $SO(1, 9)/(SO(1, 1) \times SO(8))$ .

Now we are ready to separate  $D^{A\alpha}$  into first and second class constraints in a Lorentz-covariant form. This separation is given by the equalities [111]:

$$\begin{aligned}
D^{A\alpha} &= \frac{1}{p^+} [(\sigma^a v^+)^a D_a^A + (\not{p} \sigma^+ \sigma^a v^-)^a G_a^A], & p^+ &= p^\nu u_\nu^+, \\
D^{Aa} &= (v^+ \sigma^a \not{p})_\beta D^{A\beta}, & G^{Aa} &= \frac{1}{2} (v^- \sigma^a \sigma^+)_\beta D^{A\beta}.
\end{aligned} \tag{6.159}$$

Here  $D^{Aa}$  are first class constraints and  $G^{Aa}$  are second class ones:

$$\begin{aligned}
\{D^{Aa}(\underline{\sigma}_1), D^{Bb}(\underline{\sigma}_2)\} &= -2i\delta^{AB} C^{ab} p^+ T_0 \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2) \\
\{G^{Aa}(\underline{\sigma}_1), G^{Bb}(\underline{\sigma}_2)\} &= i\delta^{AB} C^{ab} p^+ \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2).
\end{aligned}$$

It is convenient to pass from second class constraints  $G^{Aa}$  to first class constraints  $\hat{G}^{Aa}$ , without changing the actual degrees of freedom [111], [96] :

$$G^{Aa} \rightarrow \hat{G}^{Aa} = G^{Aa} + (p^+)^{1/2} \Psi^{Aa} \quad \Rightarrow \quad \{\hat{G}^{Aa}(\underline{\sigma}_1), \hat{G}^{Bb}(\underline{\sigma}_2)\} = 0,$$

where  $\Psi^{Aa}(\underline{\sigma})$  are fermionic ghosts which abelianize our second class constraints as a consequence of the Poisson bracket relation

$$\{\Psi^{Aa}(\underline{\sigma}_1), \Psi^{Bb}(\underline{\sigma}_2)\} = -i\delta^{AB}C^{ab}\delta^p(\underline{\sigma}_1 - \underline{\sigma}_2).$$

It turns out that the constraint algebra is much more simple, if we work not with  $D^{Aa}$  and  $\hat{G}^{Aa}$  but with  $\hat{T}^{A\alpha}$  given by

$$\begin{aligned}\hat{T}^{A\alpha} &= (p^+)^{-1/2}[(\sigma^a v^+)^\alpha D_a^A + (\not{p}\sigma^+ \sigma^a v^-)^\alpha \hat{G}_a^A] \\ &= (p^+)^{1/2}D^{A\alpha} + (\not{p}\sigma^+ \sigma^a v^-)^\alpha \Psi_a^A.\end{aligned}$$

After the introduction of the auxiliary fermionic variables  $\Psi^{Aa}$ , we have to modify some of the constraints, to preserve their first class property. Namely  $T_j$ ,  $I^{ab}$  and  $I^{-a}$  change as follows

$$\begin{aligned}\hat{T}_j &= T_j + \frac{i}{2}C^{ab}\sum_A \Psi_a^A \partial_j \Psi_b^A, \\ \hat{I}^{ab} &= I^{ab} + J^{ab}, \quad J^{ab} = \int d^p \sigma j^{ab}(\underline{\sigma}), \quad j^{ab} = \frac{i}{4}(v^- \sigma_c \sigma^{ab} \sigma^+ \sigma_d v^-) \sum_A \Psi^{Ac} \Psi^{Ad}, \\ \hat{I}^{-a} &= I^{-a} + J^{-a}, \quad J^{-a} = \int d^p \sigma j^{-a}(\underline{\sigma}), \quad j^{-a} = -(p^+)^{-1} j^{ab} p_b.\end{aligned}$$

As a consequence, we can write down the Hamiltonian for the considered model in the form:

$$\begin{aligned}H &= \int d^p \sigma [\lambda^0 T_0(\underline{\sigma}) + \lambda^j \hat{T}_j(\underline{\sigma}) + \sum_A \lambda^{A\alpha} \hat{T}_\alpha^A(\underline{\sigma})] + \\ &\quad \lambda_{ab} \hat{I}^{ab} + \lambda_{-+} I^{-+} + \lambda_{+a} I^{+a} + \lambda_{-a} \hat{I}^{-a}.\end{aligned}$$

The constraints entering  $H$  are all first class, irreducible and Lorentz- covariant. Their algebra reads (only the non-zero Poisson brackets are written):

$$\begin{aligned}\{T_0(\underline{\sigma}_1), \hat{T}_j(\underline{\sigma}_2)\} &= (T_0(\underline{\sigma}_1) + T_0(\underline{\sigma}_2))\partial_j \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\ \{\hat{T}_j(\underline{\sigma}_1), \hat{T}_k(\underline{\sigma}_2)\} &= (\delta_j^l \hat{T}_k(\underline{\sigma}_1) + \delta_k^l \hat{T}_j(\underline{\sigma}_2))\partial_l \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\ \{\hat{T}_j(\underline{\sigma}_1), \hat{T}_\alpha^A(\underline{\sigma}_2)\} &= (\hat{T}_\alpha^A(\underline{\sigma}_1) + \frac{1}{2}\hat{T}_\alpha^A(\underline{\sigma}_2))\partial_j \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\ \{\hat{T}_\alpha^A(\underline{\sigma}_1), \hat{T}_\beta^B(\underline{\sigma}_2)\} &= i\delta^{AB}\sigma_{\alpha\beta}^+ T_0 \delta^p(\underline{\sigma}_1 - \underline{\sigma}_2), \\ \{I^{-+}, \hat{T}_\alpha^A\} &= \frac{1}{2}\hat{T}_\alpha^A, \quad \{\hat{I}^{-a}, \hat{T}_\alpha^A\} = (2p^+)^{-1}[p^a \hat{T}_\alpha^A + (\sigma^+ \sigma^{ab} v^-)_\alpha \Psi_b^A T_0],\end{aligned}$$

$$\begin{aligned}
\{\hat{I}^{ab}, \hat{I}^{cd}\} &= C^{bc}\hat{I}^{ad} - C^{ac}\hat{I}^{bd} + C^{ad}\hat{I}^{bc} - C^{bd}\hat{I}^{ac}, \\
\{I^{-+}, I^{+a}\} &= I^{+a}, \quad \{I^{-+}, \hat{I}^{-a}\} = -\hat{I}^{-a}, \\
\{\hat{I}^{ab}, I^{+c}\} &= C^{bc}I^{+a} - C^{ac}I^{+b}, \quad \{\hat{I}^{ab}, \hat{I}^{-c}\} = C^{bc}\hat{I}^{-a} - C^{ac}\hat{I}^{-b}, \\
\{I^{+a}, \hat{I}^{-b}\} &= C^{ab}I^{-+} + \hat{I}^{ab}, \\
\{\hat{I}^{-a}, \hat{I}^{-b}\} &= -\int d^p\sigma(p^+)^{-2}j^{ab}T_0.
\end{aligned}$$

Having in mind the above algebra, one can construct the corresponding BRST charge  $\Omega$  [81] (\*=complex conjugation)

$$\Omega = \Omega^{min} + \pi_M \bar{\mathcal{P}}^M, \quad \{\Omega, \Omega\} = 0, \quad \Omega^* = \Omega, \quad (6.160)$$

where  $M = 0, j, A\alpha, ab, -+, +a, -a$ .  $\Omega^{min}$  in (6.160) can be written as

$$\begin{aligned}
\Omega^{min} &= \Omega^{brane} + \Omega^{aux}, \\
\Omega^{brane} &= \int d^p\sigma \{T_0\eta^0 + \hat{T}_j\eta^j + \sum_A \hat{T}_\alpha^A \eta^{A\alpha} + \mathcal{P}_0[(\partial_j\eta^j)\eta^0 + (\partial_j\eta^0)\eta^j] + \\
&\quad + \mathcal{P}_k(\partial_j\eta^k)\eta^j + \sum_A \mathcal{P}_\alpha^A[\eta^j\partial_j\eta^{A\alpha} - \frac{1}{2}\eta^{A\alpha}\partial_j\eta^j] - \frac{i}{2}\mathcal{P}_0\sum_A \eta^{A\alpha}\sigma_{\alpha\beta}^+\eta^{A\beta}\}, \\
\Omega^{aux} &= \hat{I}^{ab}\eta_{ab} + I^{-+}\eta_{-+} + I^{+a}\eta_{+a} + \hat{I}^{-a}\eta_{-a} \\
&\quad + (\mathcal{P}^{ac}\eta_{.c}^b - \mathcal{P}^{bc}\eta_{.c}^a + 2\mathcal{P}^{+a}\eta_+^b + 2\mathcal{P}^{-a}\eta_-^b)\eta_{ab} \\
&\quad + (\mathcal{P}^{+a}\eta_{+a} - \mathcal{P}^{-a}\eta_{-a})\eta_{-+} + (\mathcal{P}^{-+}\eta_-^a + \mathcal{P}^{ab}\eta_{-b})\eta_{+a} \\
&\quad + \frac{1}{2}\int d^p\sigma \{\sum_A \mathcal{P}_\alpha^A \eta^{A\alpha}\eta_{-+} + (p^+)^{-1}\sum_A [p^a\mathcal{P}_\alpha^A - (\sigma^+\sigma^{ab}v^-)_\alpha\Psi_b^A\mathcal{P}_0]\eta^{A\alpha}\eta_{-a} \\
&\quad - (p^+)^{-2}j^{ab}\mathcal{P}_0\eta_{-b}\eta_{-a}\}.
\end{aligned}$$

These expressions for  $\Omega^{brane}$  and  $\Omega^{aux}$  show that we have found a set of constraints which ensure the first rank property of the model.

$\Omega^{min}$  can be represented also in the form

$$\begin{aligned}
\Omega^{min} &= \int d^p\sigma [(T_0 + \frac{1}{2}T_0^{gh})\eta^0 + (\hat{T}_j + \frac{1}{2}T_j^{gh})\eta^j + \sum_A (\hat{T}_\alpha^A + \frac{1}{2}T_\alpha^{Agh})\eta^{A\alpha}] \\
&\quad + (\hat{I}^{ab} + \frac{1}{2}I_{gh}^{ab})\eta_{ab} + (I^{-+} + \frac{1}{2}I_{gh}^{-+})\eta_{-+} + (I^{+a} + \frac{1}{2}I_{gh}^{+a})\eta_{+a} + (\hat{I}^{-a} + \frac{1}{2}I_{gh}^{-a})\eta_{-a} \\
&\quad + \int d^p\sigma \partial_j \left( \frac{1}{2}\mathcal{P}_k\eta^k\eta^j + \frac{1}{4}\sum_A \mathcal{P}_\alpha^A \eta^{A\alpha}\eta^j \right).
\end{aligned}$$

Here a super(sub)script  $gh$  is used for the ghost part of the total gauge generators

$$G^{tot} = \{\Omega, \mathcal{P}\} = \{\Omega^{min}, \mathcal{P}\} = G + G^{gh}.$$

We recall that the Poisson bracket algebras of  $G^{tot}$  and  $G$  coincide for first rank systems. The manifest expressions for  $G^{gh}$  are:

$$\begin{aligned}
T_0^{gh} &= 2\mathcal{P}_0\partial_j\eta^j + (\partial_j\mathcal{P})\eta^j, \\
T_j^{gh} &= 2\mathcal{P}_0\partial_j\eta^0 + (\partial_j\mathcal{P}_0)\eta^0 + \mathcal{P}_j\partial_k\eta^k + \mathcal{P}_k\partial_j\eta^k + (\partial_k\mathcal{P}_j)\eta^k \\
&\quad + \frac{3}{2}\sum_A\mathcal{P}_\alpha^A\partial_j\eta^{A\alpha} + \frac{1}{2}\sum_A(\partial_j\mathcal{P}_\alpha^A)\eta^{A\alpha}, \\
T_\alpha^{Agh} &= -\frac{3}{2}\mathcal{P}_\alpha^A\partial_j\eta^j - (\partial_j\mathcal{P}_\alpha^A)\eta^j - i\mathcal{P}_0\sigma_{\alpha\beta}^+\eta^{A\beta} + \\
&\quad + \frac{1}{2}\mathcal{P}_\alpha^A\eta_{-+} + (2p^+)^{-1}[p^a\mathcal{P}_\alpha^A - (\sigma^+\sigma^{ab}v^-)_\alpha\Psi_b^A\mathcal{P}_0]\eta_{-a}, \\
I_{gh}^{ab} &= 2(\mathcal{P}^{ac}\eta_{.c}^b - \mathcal{P}^{bc}\eta_{.c}^a) + (\mathcal{P}^{+a}\eta_+^b - \mathcal{P}^{+b}\eta_+^a) + (\mathcal{P}^{-a}\eta_-^b - \mathcal{P}^{-b}\eta_-^a), \\
I_{gh}^{+-} &= \mathcal{P}^{+a}\eta_{+a} - \mathcal{P}^{-a}\eta_{-a} + \frac{1}{2}\int d^p\sigma\sum_A\mathcal{P}_\alpha^A\eta^{A\alpha}, \\
I_{gh}^{+a} &= 2\mathcal{P}^{+b}\eta_{.b}^a - \mathcal{P}^{+a}\eta_{-+} + \mathcal{P}^{-+}\eta_-^a + \mathcal{P}^{ab}\eta_{-b}, \\
I_{gh}^{-a} &= 2\mathcal{P}^{-b}\eta_{.b}^a + \mathcal{P}^{-a}\eta_{-+} - \mathcal{P}^{-+}\eta_+^a + \mathcal{P}^{ab}\eta_{+b} + \\
&\quad + \int d^p\sigma\{(2p^+)^{-1}\sum_A[p^a\mathcal{P}_\alpha^A - (\sigma^+\sigma^{ab}v^-)_\alpha\Psi_b^A\mathcal{P}_0]\eta^{A\alpha} - (p^+)^{-2}j^{ab}\mathcal{P}_0\eta_{-b}\}.
\end{aligned}$$

Up to now, we introduced canonically conjugated ghosts  $(\eta^M, \mathcal{P}_M)$ ,  $(\bar{\eta}_M, \bar{\mathcal{P}}^M)$  and momenta  $\pi_M$  for the Lagrange multipliers  $\lambda^M$  in the Hamiltonian. They have Poisson brackets and Grassmann parity as follows ( $\epsilon_M$  is the Grassmann parity of the corresponding constraint):

$$\begin{aligned}
\{\eta^M, \mathcal{P}_N\} &= \delta_N^M, & \epsilon(\eta^M) &= \epsilon(\mathcal{P}_M) = \epsilon_M + 1, \\
\{\bar{\eta}_M, \bar{\mathcal{P}}^N\} &= -(-1)^{\epsilon_M\epsilon_N}\delta_M^N, & \epsilon(\bar{\eta}_M) &= \epsilon(\bar{\mathcal{P}}^M) = \epsilon_M + 1, \\
\{\lambda^M, \pi_N\} &= \delta_N^M, & \epsilon(\lambda^M) &= \epsilon(\pi_M) = \epsilon_M.
\end{aligned}$$

The BRST-invariant Hamiltonian is

$$H_{\tilde{\chi}} = H^{min} + \{\tilde{\chi}, \Omega\} = \{\tilde{\chi}, \Omega\}, \quad (6.161)$$

because from  $H_{canonical} = 0$  it follows that  $H^{min} = 0$ . In this formula  $\tilde{\chi}$  stands for the gauge fixing fermion ( $\tilde{\chi}^* = -\tilde{\chi}$ ). We use the following representation for the latter

$$\tilde{\chi} = \chi^{min} + \bar{\eta}_M(\chi^M + \frac{1}{2}\rho_{(M)}\pi^M), \quad \chi^{min} = \lambda^M\mathcal{P}_M,$$

where  $\rho_{(M)}$  are scalar parameters and we have separated the  $\pi^M$ -dependence from  $\chi^M$ . If we adopt that  $\chi^M$  does not depend on the ghosts  $(\eta^M, \mathcal{P}_M)$  and  $(\bar{\eta}_M, \bar{\mathcal{P}}^M)$ , the Hamiltonian  $H_{\tilde{\chi}}$  from (6.161) takes the form

$$\begin{aligned} H_{\tilde{\chi}} &= H_{\chi}^{min} + \mathcal{P}_M \bar{\mathcal{P}}^M - \pi_M (\chi^M + \frac{1}{2} \rho_{(M)} \pi^M) + \\ &+ \bar{\eta}_M \left[ \{\chi^M, G_N\} \eta^N + \frac{1}{2} (-1)^{\epsilon_N} \mathcal{P}_Q \{\chi^M, U_{NP}^Q\} \eta^P \eta^N \right], \end{aligned} \quad (6.162)$$

where

$$H_{\chi}^{min} = \{\chi^{min}, \Omega^{min}\},$$

and generally  $\{\chi^M, U_{NP}^Q\} \neq 0$  as far as the structure coefficients of the constraint algebra  $U_{NP}^M$  depend on the phase-space variables.

One can use the representation (6.162) for  $H_{\tilde{\chi}}$  to obtain the corresponding BRST invariant Lagrangian

$$L_{\tilde{\chi}} = L + L_{GH} + L_{GF}.$$

Here  $L_{GH}$  stands for the ghost part and  $L_{GF}$  for the gauge fixing part of the Lagrangian. If one does not intend to pass to the Lagrangian formalism, one may restrict oneself to the minimal sector  $(\Omega^{min}, \chi^{min}, H_{\chi}^{min})$ . In particular, this means that Lagrange multipliers are not considered as dynamical variables anymore. With this particular gauge choice,  $H_{\chi}^{min}$  is a linear combination of the total constraints

$$\begin{aligned} H_{\chi}^{min} &= H_{brane}^{min} + H_{aux}^{min} = \\ &= \int d^p \sigma \left[ \Lambda^0 T_0^{tot}(\underline{\sigma}) + \Lambda^j T_j^{tot}(\underline{\sigma}) + \sum_A \Lambda^{A\alpha} T_{\alpha}^{Atot}(\underline{\sigma}) \right] + \\ &+ \Lambda_{ab} I_{tot}^{ab} + \Lambda_{-+} I_{tot}^{-+} + \Lambda_{+a} I_{tot}^{+a} + \Lambda_{-a} I_{tot}^{-a}, \end{aligned}$$

and we can treat here the Lagrange multipliers  $\Lambda^0, \dots, \Lambda_{-a}$  as constants. Of course, this does not fix the gauge completely.

### 6.3 Comments

To ensure that the harmonics and their conjugate momenta are pure gauge degrees of freedom, we have to consider as physical observables only such functions

on the phase space which do not carry any  $SO(1,1) \times SO(8)$  indices. More precisely, these functions are defined by the following expansion

$$F(y, u, v; p_y, p_u, p_v) = \sum [u_{\nu_1}^{a_1} \dots u_{\nu_k}^{a_k} p_{u\nu_{k+1}}^{a_{k+1}} \dots p_{u\nu_{k+l}}^{a_{k+l}}]_{SO(8) \text{ singlet}} \\ v_{\alpha_1}^+ \dots v_{\alpha_m}^+ v_{\alpha_{m+1}}^- \dots v_{\alpha_{m+n}}^- p_v^{+\beta_1} \dots p_v^{+\beta_r} p_v^{-\beta_{r+1}} \dots p_v^{-\beta_{m-n+r}} \\ F_{\beta_1 \dots \beta_{m-n+r}}^{\alpha_1 \dots \alpha_{m+n}, \nu_1 \dots \nu_{k+l}}(y, p_y),$$

where  $(y, p_y)$  are the non-harmonic phase space conjugated pairs.

## 7 CONCLUSIONS

The dissertation is devoted to the description and further investigation of the properties of *null*  $p$ -branes. The necessary preliminaries are given in the Introduction. Then we explain how the above extended objects arise in the context of string theory. The following section consider the known classical and quantum properties of the tensionless branes. The original results are described in sections 4-6.

In the forth section we perform BRST quantization of the null bosonic  $p$ -branes using four different types of operator ordering. It is shown that one can or can not obtain critical dimension for the null string ( $p = 1$ ), depending on the choice of the operator ordering and corresponding vacuum states. When  $p > 1$ , operator orderings leading to critical dimension in the  $p = 1$  case are forbidden by the Jacobi identities. Admissible orderings give no restrictions on the dimension of the embedding space-time. This is connected with the fact that the full constraint algebra has no nontrivial central extension, but there are  $p$  subalgebras which possess non-trivial central extensions. When  $p = 1$ , there is one such subalgebra and it coincides with the full algebra. This section is based on the paper [71].

In the fifth section we perform some investigation on the classical dynamics of the null bosonic branes in curved background. We write down the action, prove its reparametrization invariance and give the equations of motion and constraints in an arbitrary gauge. Then we construct the corresponding Hamiltonian and compute the constraint algebra. In the following subsection we consider the dynamics of null membranes ( $p = 2$ ) in a four dimensional, stationary, axially symmetric gravity background. Some exact solutions of the equations of motion and of the constraints are found there. The next subsection is devoted to examples of such solutions in Minkowski, de Sitter, Schwarzschild, Taub-NUT and Kerr space-times. This section is based on the paper [74].

In the sixth section we consider a model for tensionless super  $p$ -branes with  $N$  global chiral supersymmetries in 10-dimensional Minkowski space-time. We show that the action is reparametrization and  $\kappa$ -invariant. After establishing the symme-

tries of the action, we give the general solution of the classical equations of motion in a particular gauge. In the case of null superstrings ( $p=1$ ) we find the general solution in an arbitrary gauge. Starting with a Hamiltonian which is a linear combination of first and mixed (first and second) class constraints, we succeed to obtain a new one, which is a linear combination of first class, BFV-irreducible and Lorentz-covariant constraints only. This is done with the help of the introduced auxiliary harmonic variables. Then we give manifest expressions for the classical BRST charge, the corresponding total constraints and BRST-invariant Hamiltonian. It turns out that in the given formulation our model is a first rank dynamical system. This section is based on the papers [67, 68, 69].



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## 9 Appendix A

Here we briefly comment on the possible central extensions of the algebra, given by the commutators:

$$\begin{aligned} [A_{\underline{n}}, A_{\underline{m}}] &= 0, \\ [A_{\underline{n}}, B_{j,\underline{m}}] &= (n_j - m_j)A_{\underline{n}+\underline{m}}, \\ [B_{j,\underline{n}}, B_{k,\underline{m}}] &= (\delta_j^l n_k - \delta_k^l m_j)B_{l,\underline{n}+\underline{m}} \quad , \quad (j, k = 1, 2, \dots, p). \end{aligned}$$

To begin with, we modify the right hand sides of the above equalities as follows:

$$\begin{aligned} [A_{\underline{n}}, A_{\underline{m}}] &= d(\underline{n}, \underline{m}) \\ [A_{\underline{n}}, B_{j,\underline{m}}] &= (n_j - m_j)A_{\underline{n}+\underline{m}} + d_j(\underline{n}, \underline{m}), \\ [B_{j,\underline{n}}, B_{k,\underline{m}}] &= (\delta_j^l n_k - \delta_k^l m_j)B_{l,\underline{n}+\underline{m}} + d_{jk}(\underline{n}, \underline{m}). \end{aligned}$$

Checking the Jacobi identities, involving the triplets  $(A, A, B)$ ,  $(A, B, B)$  and  $(B, B, B)$ , one shows that there are only trivial solutions for  $d(\underline{n}, \underline{m})$ ,  $d_j(\underline{n}, \underline{m})$  and  $d_{jk}(\underline{n}, \underline{m})$ . Namely,

$$\begin{aligned} d(\underline{n}, \underline{m}) &= 0 \quad , \quad d_j(\underline{n}, \underline{m}) = (n_j - m_j)f(\underline{n} + \underline{m}), \\ d_{jk}(\underline{n}, \underline{m}) &= (\delta_j^l n_k - \delta_k^l m_j)g_l(\underline{n} + \underline{m}), \end{aligned}$$

where  $f, g_j$  are arbitrary functions of their arguments. In particular, there exist the solutions

$$\begin{aligned} d_j(\underline{n}, \underline{m}) &= 2\alpha n_j \delta_{\underline{n}+\underline{m}, \underline{0}} \quad , \quad \alpha = \text{const}, \\ d_{jk}(\underline{n}, \underline{m}) &= (\beta_j n_k + \beta_k n_j) \delta_{\underline{n}+\underline{m}, \underline{0}} \quad , \quad \beta_j = \text{const}, \end{aligned}$$

which might appear because of the operator ordering in  $A_{\underline{n}}$  and  $B_{j,\underline{n}}$ . However, there are  $p$  subalgebras with non-trivial central extensions (no summation over  $j$ ):

$$\begin{aligned} [A_{\underline{n}}, A_{\underline{m}}] &= 0, \\ [A_{\underline{n}}, B_{j,\underline{m}}] &= (n_j - m_j)A_{\underline{n}+\underline{m}} + (q_j n_j^2 + r_j)n_j \delta_{\underline{n}+\underline{m}, \underline{0}}, \\ [B_{j,\underline{n}}, B_{j,\underline{m}}] &= (n_j - m_j)B_{j,\underline{n}+\underline{m}} + (s_j n_j^2 + t_j)n_j \delta_{\underline{n}+\underline{m}, \underline{0}}, \quad q_j, r_j, s_j, t_j = \text{const}. \end{aligned}$$

When  $p = 1$ , there is one such subalgebra and it coincides with the full algebra.

## 10 Appendix B

We briefly describe here our 10-dimensional conventions. Dirac  $\gamma$ - matrices obey

$$\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\eta_{\mu\nu}$$

and are taken in the representation

$$\Gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\dot{\alpha}}^{\beta} \\ (\tilde{\sigma}^\mu)^{\beta}_{\dot{\alpha}} & 0 \end{pmatrix} .$$

$\Gamma^{11}$  and charge conjugation matrix  $C_{10}$  are given by

$$\Gamma^{11} = \Gamma^0 \Gamma^1 \dots \Gamma^9 = \begin{pmatrix} \delta_{\alpha}^{\beta} & 0 \\ 0 & -\delta_{\dot{\alpha}}^{\dot{\beta}} \end{pmatrix} ,$$

$$C_{10} = \begin{pmatrix} 0 & C^{\alpha\dot{\beta}} \\ (-C)^{\dot{\alpha}\beta} & 0 \end{pmatrix} ,$$

and the indices of right and left Majorana-Weyl fermions are raised as

$$\psi^{\alpha} = C^{\alpha\dot{\beta}} \psi_{\dot{\beta}} \quad , \quad \phi^{\dot{\alpha}} = (-C)^{\dot{\alpha}\beta} \phi_{\beta} .$$

We use  $D = 10$   $\sigma$ -matrices with undotted indices

$$(\sigma^{\mu})^{\alpha\beta} = C^{\alpha\dot{\alpha}} (\tilde{\sigma}^{\mu})_{\dot{\alpha}}^{\beta} \quad , \quad (\sigma^{\mu})_{\alpha\beta} = (-C)^{-1}_{\dot{\beta}\dot{\alpha}} (\sigma^{\mu})_{\dot{\alpha}}^{\dot{\beta}} \quad ,$$

and the notation

$$\sigma^{\mu_1 \dots \mu_n} \equiv \sigma^{[\mu_1 \dots \mu_n]}$$

for their antisymmetrized products.

From the corresponding properties of  $D = 10$   $\gamma$ -matrices, it follows that

$$\begin{aligned} (\sigma^{\mu})_{\alpha\gamma} (\sigma^{\nu})^{\gamma\beta} + (\sigma^{\nu})_{\alpha\gamma} (\sigma^{\mu})^{\gamma\beta} &= -2\delta_{\alpha}^{\beta} \eta^{\mu\nu} \quad , \\ (\sigma_{\mu_1 \dots \mu_{2s+1}})^{\alpha\beta} &= (-1)^s (\sigma_{\mu_1 \dots \mu_{2s+1}})^{\beta\alpha} \quad , \\ \sigma^{\mu} \sigma^{\nu_1 \dots \nu_n} &= \sigma^{\mu\nu_1 \dots \nu_n} + \sum_{k=1}^n (-1)^k \eta^{\mu\nu_k} \sigma^{\nu_1 \dots \nu_{k-1} \nu_{k+1} \dots \nu_n} . \end{aligned}$$

The Fierz identity for the  $\sigma$ -matrices reads:

$$(\sigma_{\mu})^{\alpha\beta} (\sigma^{\mu})^{\gamma\delta} + (\sigma_{\mu})^{\beta\gamma} (\sigma^{\mu})^{\alpha\delta} + (\sigma_{\mu})^{\gamma\alpha} (\sigma^{\mu})^{\beta\delta} = 0 .$$

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